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Complex cobordism and the elliptic genus

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Abstract. These are notes from the special lecture given by Professor F. Hirzebruch during the Algebraic Geometry Conference Hirzebruch 70. The lecture surveyed the Krichever–Höhn elliptic genus ([K], [Hö]) and reported about the work of Totaro [T] who characterized the genus by its invariance under flops (interchanging certain small resolutions). It was shown that this invariance leads to a differential-functional equation for the characteristic power series of the genus and that this differential equation (related to the Lamé equation) characterizes the genus.

1. Complex cobordism

1.1. Let X be a compact complex manifold of (complex) dimension n and let $c_i \in H^{2i}(X, \mathbb{Z})$ denote the Chern classes of the complex tangent bundle of X. Then the numbers $c_{\lambda_1} c_{\lambda_2} \dots c_{\lambda_r}[X]$ (this denotes $c_{\lambda_1} c_{\lambda_2} \dots c_{\lambda_r}$ evaluated on the fundamental cycle of X), where $\lambda_1 + \cdots + \lambda_r = n$, are called the *Chern numbers* of the manifold X. In particular,

$$c_n[X] = e(X), \tag{1.1.1}$$

where $e(X)$ is the topological Euler characteristic of X. By definition, the Chern classes depend only on the complex tangent bundle. Therefore, they are defined for an almost complex manifold X. This is a compact smooth real manifold of dimension $2n$ together with an n-dimensional complex vector bundle whose underlying real bundle is the real tangent bundle TX of X. Then X has a natural orientation and (1.1.1) is still true. A stable almost complex structure on X is given by a complex vector bundle over X whose underlying real bundle is TX plus a trivial bundle. Such a manifold is orientable, but does not have a natural orientation. We always choose an orientation and then speak of a stable almost complex manifold. Chern classes and Chern numbers are well defined. In general, the equation (1.1.1) is not true anymore. In view of the results of Milnor [M] and Novikov [N]

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we can introduce the complex cobordism ring Ω_*^U generated by all stable almost complex manifolds (disjoint union and cartesian product as ring operations) where two manifolds determine the same element in Ω_*^U if and only if their Chern numbers coincide. The "negative" of a manifold is simply given by changing the orientation. The ring Ω_*^U is a polynomial ring over $\mathbb{Z}$. However, for $\Omega_*^U \otimes \mathbb{Q}$ a base sequence X_1, X_2,..., where $\dim_{\mathbb{C}} X_i = i$ is easier to describe. For example, $\mathbb{CP}^1$, $\mathbb{CP}^2$, $\mathbb{CP}^3$,... is a base sequence.

Write the total Chern class of the manifold X in the form

$$c(X) = 1 + c_1 + c_2 + ... + c_n = (1 + x_1)(1 + x_2)...(1 + x_n),$$

where the x_i are formal symbols (roots) treated as elements of the second cohomology group of some extension of the cohomology ring of X. Then every symmetric expression in the x_i can be written in terms of the c_i.

The sequence of varieties X_1, X_2,... of stable almost complex manifolds is a base sequence for the cobordism ring $\Omega_*^U \otimes \mathbb{Q}$ if and only if $s_n[X_n] \neq 0$, where $s_n = x_1^n + ... + x_n^n$.

1.2. Examples of stable almost complex manifolds.

(1) Let $g \in H^2(\mathbb{CP}^n, \mathbb{Z})$ be a cohomological generator corresponding to the complex hyperplane $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$. Then

$$c(\mathbb{CP}^n) = (1 + g)^{n+1}.$$

Let a, b be positive integers such that $a + b = n + 1$.

Let $T_{\mathbb{P}^n}$ be the complex tangent bundle of $\mathbb{CP}^n$ $(= \mathbb{P}^n)$ and $\mathcal{O}_{\mathbb{P}^n}$ the trivial complex line bundle. By $\mathcal{O}_{\mathbb{P}^n}(1)$ we denote as usual the complex line bundle with first Chern class g. We have an isomorphism $T_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n} \simeq (n+1)\mathcal{O}_{\mathbb{P}^n}(1)$ as complex vector bundles. We introduce another complex structure on the complex vector bundle $(n + 1)\mathcal{O}_{\mathbb{P}^n}(1)$ (of fibre dimension $n + 1$) by conjugation on the first a components (i.e., we take $a\overline{\mathcal{O}_{\mathbb{P}^n}(1)} \oplus b\mathcal{O}_{\mathbb{P}^n}(1)$). In this way we define a stable almost complex structure on $\mathbb{CP}^n$ which will be denoted by $\mathbb{CP}^{a,b}$. Clearly,

$$c(\mathbb{CP}^{a,b}) = (1 - g)^a(1 + g)^b.$$

The first Chern class vanishes if and only if $a = b$. To define Chern numbers we use the standard orientation given by $g^n[\mathbb{CP}^{a,b}] = 1$. In particular, $c(\mathbb{CP}^{2,2}) = 1 - 2g^2$, so we get a 3-dimensional projective space with a stable almost complex structure with all Chern numbers equal to 0. Let us write $\tilde{\mathbb{P}}^3 = \mathbb{CP}^{2,2}$.

(2) Let us also introduce a stable almost complex structure on the smooth 4-dimensional quadric by inducing a stable almost complex structure from $\mathbb{CP}^{2,4}$. The obtained manifold will be denoted by $\tilde{Q}^4$. Clearly,

$$c(\tilde{Q}^4) = (1 - g)^2(1 + g)^4(1 + 2g)^{-1} = 1 - g^2 - 2g^3 + 3g^4,$$

so the first Chern class of $\tilde{Q}^4$ is 0. In fact, this is an almost complex structure. The equation (1.1.1) checks because $g^4[\tilde{Q}^4] = 2$ and $e(\tilde{Q}^4) = 6$.

For these examples see [Hö].

1.3. Let $\mathcal{R}$ be a ring containing $\mathbb{Q}$. A ring homomorphism $\Omega_*^U \otimes \mathbb{Q} \to \mathcal{R}$ is by definition called a genus. This definition appeared already in [H]. Any genus depends only on Chern numbers, since two manifolds represent the same class in Ω_*^U if and only if all their Chern numbers coincide.

All genera can be constructed from a power series in the following way.

Let $Q(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ be a formal power series in x with coefficients in $\mathcal{R}$. Then the product $Q(x_1)Q(x_2)\ldots Q(x_n)$, where x_i are the formal roots of $c(X)$, can be expressed in terms of c_i and we can evaluate the part lying in $H^{2n}(X,\mathbb{Z})$ on the fundamental cycle of X. In this way one gets a well defined genus $K_Q\colon \Omega_*^U \otimes \mathbb{Q} \to \mathcal{R}$ associated to the power series Q.

Every genus can be given by precisely one normalized power series ($a_0 = 1$). The genus K_Q is given by the normalized power series $Q(a_0 x)/a_0$ if $a_0 \neq 0$. The genus K_Q/a_0^n is given by the normalized power series $Q(x)/a_0$. If $a_0 = 0$, then the genus is $a_1^n c_n[X]$. The genus $c_n[X]$ belongs to the power series $1 + x$, but also to the power series x.

1.4. Riemann–Roch–Hirzebruch theorem.

For a holomorphic vector bundle W over a compact complex manifold X of complex dimension n the holomorphic Euler number

$$\chi(X,W) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}} H^i(X, \mathcal{O}(W))$$

is defined, where $\mathcal{O}(W)$ is the sheaf of local holomorphic sections of W. Then

$$(RRH) \qquad \chi(X,W) = \left(ch(W) \prod_{i=1}^{n} \frac{x_i}{1-e^{-x_i}} \right) [X],$$

where the x_i are the formal roots of the total Chern class of X. Moreover, $ch(W)$ is the Chern character of W:

$$ch(W) = e^{y_1} + \ldots + e^{y_m},$$

where the fibre dimension of W is m and the y_i are the formal roots of the total Chern class of W. The formula (RRH) was proved in [H] if X is projective algebraic and follows from the Atiyah–Singer index theorem [AS] if X is a complex manifold.

If W is associated to the tangent bundle of X by a representation of $GL(n,\mathbb{C})$, then the Chern classes of W can be expressed by those of the tangent bundle of X (Borel–Hirzebruch). Then $\chi(X,W)$ can be defined if X is an element of Ω_{2n}^U, because only the Chern numbers enter in the right hand side of (RRH). It is known that $\chi(X,W)$ is an integer.

If T is the tangent bundle of X and T^* its dual, then this remark applies to any polynomial in the exterior or symmetric powers of T and T^*. Exterior and symmetric powers are denoted by $\bigwedge^p$ and S^p respectively.

1.5. The χ_y-genus and the Krichever–Höhn elliptic genus.

Set $\chi^p = \chi(X, \bigwedge^p T^*)$ and $\chi_y = \sum_p \chi^p y^p$. For a Kähler manifold the space $H^q(X, \mathcal{O}(\bigwedge^p T^*))$ is isomorphic to the space of harmonic forms of type (p,q) whose dimension is denoted by $h^{p,q}$ and

$$\chi_y = \sum_{p,q} (-1)^q h^{p,q} y^p$$

which for $y = 0$ is the arithmetic genus (Todd genus), for $y = -1$ the Euler number and for $y = 1$ the signature by the Hodge signature theorem. We can study more generally

$$\chi_y(X,K^r)=\sum_{p=0}^{n}\chi^p(X,K^r)y^p$$

with $\chi^p(X,K^r)=\chi(X,\bigwedge^p T^*\otimes K^r)$, where $K=\bigwedge^n T^*$ is the canonical line bundle. Then, for every r, the polynomial $\chi_y(X,K^r)$ is the genus belonging to the power series

$$Q(x)=\frac{x}{1-e^{-x}}(1+ye^{-x})e^{-rx} \tag{1.5.1}$$

(we may regard r as an indeterminate).

For $y=-1$ the genus equals $c_n[X]$ for all r.

For $y=0$ it is $\chi(X,K^r)$ which is (for a complex manifold) the r-th plurigenus if the higher cohomology groups vanish ($r\in\mathbb{Z}$, $r\geq 2$).

Consider the following "extension"of the power series in (1.5.1)

$$x\cdot e^{-rx}\prod_{n=0}^{\infty}\frac{1+yq^ne^{-x}}{1-q^ne^{-x}}\prod_{n=1}^{\infty}\frac{1+y^{-1}q^ne^{x}}{1-q^ne^{x}}. \tag{1.5.2}$$

Set $\bigwedge_z W=\sum\bigwedge^p W\cdot z^p$ and $S_zW=\sum S^pW\cdot z^p$. Then the genus corresponding to this power series can be described by the following formula:

$$\varphi(X)=\chi(X,K^r\prod_{n=0}^{\infty}\bigwedge\nolimits_{yq^n}T^*\prod_{n=1}^{\infty}\bigwedge\nolimits_{y^{-1}q^n}T\prod_{n=1}^{\infty}S_{q^n}(T^*+T)).$$

In the language of physicists (Witten) this is the equivariant χ_y-genus of the loop space of X (for $r=0$ and with $q\in S^1$). Right now q is an indeterminant and $\varphi(X)$ is a power series in q which for $q=0$ equals $\chi_y(X,K^r)$. The power series for $\varphi(X)$ given above is not normalized. We have

$$a_0=(1+y)\prod_{n=1}^{\infty}\frac{(1+yq^n)(1+y^{-1}q^n)}{(1-q^n)^2}.$$

We put $y=-e^{-v}$. Then a_0 goes over into

$$\Upsilon(v)=(1-e^{-v})\prod_{n=1}^{\infty}\frac{(1-e^{-v}q^n)(1-e^{v}q^n)}{(1-q^n)^2}. \tag{1.5.3}$$

We have $\Upsilon(v)=e^{-v/2}\cdot\frac{\Theta(v)}{\Theta'(0)}$, where $\Theta(v)$ is the classical Jacobi theta function (see [Z], (6)) We now put $q=e^{2\pi i\tau}$ with τ in the upper half plane. We denote the normalized genus $\varphi(X)/a_0^n$ by $\varphi_{ell}(X)$ (for $\dim_{\mathbb{C}}X=n$). Its normalized power series equals

$$Q_{ell}(x)=xe^{-rx}\frac{\Upsilon(x+v)}{\Upsilon(x)\Upsilon(v)}.$$

Then

$$Q_{ell}(x)=xe^{-rx}F_\tau(x,v) \tag{1.5.4}$$

where

$$F_\tau(x,v)=\frac{\Theta'(0)\Theta(x+v)}{\Theta(x)\Theta(v)}$$

is the function studied in section 3 in [Z].

It will be convenient for us to quote from [Z] when elliptic and modular properties of $F_\tau(u,v)$ are needed. The coefficient a_1 in $Q_{ell}(x)$ equals $\zeta(v) - \eta_1 \frac{v}{2\pi i} - r$ where $\zeta(v)$ is the Weierstrass ζ-function for the lattice $L = 2\pi i(\mathbb{Z}\tau + \mathbb{Z})$ and $\zeta(v+2\pi i) = \zeta(v) + \eta_1$. If we choose r in such a way that a_1 is zero, then

$$F_\tau(x,v) = e^{-\zeta(v)x} \cdot \frac{\sigma(x+v)}{\sigma(x)\sigma(v)} \tag{1.5.5}$$

where $\sigma(v)$ is the Weierstrass σ-function for the lattice L. This is a function used in [K]. For a manifold with vanishing first Chern class the genus is independent of r.

2. Elliptic genera – properties

2.1. Elliptic genera – the real case.

Let Ω_*^{SO} be the cobordism ring of real oriented manifolds. Then $\Omega_*^{SO} \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \mathbb{CP}^6, \ldots]$ and similarly as above one introduces a definition of a genus $\varphi\colon \Omega_*^{SO} \otimes \mathbb{Q} \to \mathcal{R}$. In this case one says that φ is *an elliptic genus* if its "logarithm" $g(u) = \sum_{n=0}^{\infty} \frac{\varphi(\mathbb{CP}^{2n})}{2n+1} u^{2n+1}$ is given by an elliptic integral of the first kind, i.e.,

$$g'(u) = R(u)^{-\frac{1}{2}},$$

where $R(u) = 1 - 2\delta u^2 + \epsilon u^4$ $(\delta, \epsilon \in \mathcal{R})$. The theory of elliptic genera was developed by Ochanine, Landweber, Stong, Witten and others (see [O], [L], [LS], [HBJ]). Elliptic genera in the real case can be characterized as those genera which vanish on $\mathbb{CP}^{2m+1}$-bundles over closed oriented spin manifolds (see [O]). Actually $\mathbb{CP}^3$-bundles are enough for the characterization as follows from 2.2. The power series (1.5.2) and (1.5.4) are even functions for $r = 0$ and $v = \pi i$ (i.e., $y = 1$, the signature case). Then the genus φ_{ell} depends only on the Pontrjagin numbers and gives exactly the elliptic genus in the real case. We come back to this later.

2.2. Elliptic genera – the complex case.

The elliptic genus φ_{ell} satisfies the following multiplicativity property ([K], [Hö], [HBJ]).

THEOREM 2.2.1. *Let $M \longrightarrow B$ be a fibre bundle with a compact stable almost complex manifold F as a fibre and a compact connected Lie group as structure group (preserving the stable almost complex structure). Also B is assumed to be stable almost complex. Then M inherits naturally a stable almost complex structure. If $c_1(F) = 0$, then*

$$\varphi_{ell}(M) = \varphi_{ell}(F) \cdot \varphi_{ell}(B).$$

For any genus the multiplicativity for all bundles $M \to B$ as in the above theorem with given fibre F is equivalent to the constancy of the equivariant genus (see 4.2) for any circle action on F. Such rigidity or multiplicativity theorems are sophisticated and have a long history (compare [L], [HBJ]). The first proofs in the case of the elliptic genus in the real case (2.1) were given by Taubes and Bott–Taubes [BT]. Here one assumes that the fibre has a spin structure. In the above theorem the χ_y-genus is multiplicative without assumption $c_1(F) = 0$. This is essentially due to Lusztig and Kosniowski (compare [AH], p. 25). The rigidity theorems were motivated by Witten through physics. The proof of Theorem 2.2.1

(see [Hö]) uses the rigidity of the level N genera if F has a first Chern class divisible by N (see [HBJ]). This goes along the lines of [BT]. Krichever [K] has a different proof.

We choose a base sequence for the complex cobordism ring $\Omega_*^U \otimes \mathbb{Q}$ beginning with $\mathbb{CP}^1$, a $K3$-surface, the sphere S^6, and the quadric $\tilde{Q}^4$. Höhn [Hö] shows that we can continue this base sequence by the total spaces of fibrations in the sense of the above theorem with fibres $F = \mathbb{CP}^{n,n}$ (see 1.2). Then $c_1(F) = 0$ and $c(F) = (1 - g^2)^n$, all Chern numbers of F vanish. Hence $\varphi_{ell}(F) = 0$ and $\varphi_{ell} = 0$ for all total spaces of the fibrations. Thus Höhn has a base sequence X_i with $\varphi_{ell}(X_i) = 0$ for $i \geq 5$. Also Totaro [T] constructs such a base sequence, but shows that one can use for $i \geq 5$ fibrations with $\tilde{\mathbb{P}}^3 = \mathbb{CP}^{2,2}$ as a fibre.

We have the following table

	$\chi_y(X, K^r)$	φ_{ell}
$\mathbb{P}^1$	$-2r + 1 - y$	A
K3-surface	$2 - 20y + 2y^2$	B
S^6	$-y + y^2$	C
$\tilde{Q}^4$	$-y + 4y^2 - y^3$	D

Dividing χ_y by $(1 + y)^n$ gives the normalized χ_y-genus. For all the remaining elements in the chosen base sequence $\chi_y(X, K^r)$ and $\varphi_{ell}(X)$ are zero. Thus φ_{ell} depends on at most 4 parameters A, B, C and D. In fact, in the definition of φ_{ell} we have the parameters r, q, v. But one can pass from A, B, C, D to λA, $\lambda^2 B$, $\lambda^3 C$, $\lambda^4 D$ and from a lattice $2\pi i(\mathbb{Z} + \tau\mathbb{Z})$ to a multiple.

We can define φ_{ell} by its values A, B, C, D on $\mathbb{CP}^1$, a $K3$-surface, S^6, $\tilde{Q}^4$, respectively. Then φ_{ell} is a surjection onto a polynomial ring

$$\varphi_{ell}\colon \Omega_*^U \otimes \mathbb{Q} \to \mathbb{Q}[A, B, C, D].$$

This genus is universal: For every genus $\psi\colon \Omega_*^U \otimes \mathbb{Q} \to \mathcal{R}$ satisfying the multiplicativity in Theorem 2.2.1, there exists a uniquely defined ring homomorphism

$$\alpha\colon \mathbb{Q}[A, B, C, D] \to \mathcal{R}$$

with $\psi = \alpha\varphi_{ell}$. For more details see the table in 5.2.

3. Invariance of elliptic genera under flops

3.1. *The Atiyah flop*

Let us consider the 3-dimensional quadric Q given by the equation

$$\phi_1\phi_2 - \phi_3\phi_4 = 0$$

in homogeneous coordinates $[\phi_1, ..., \phi_5]$ of $\mathbb{P}^4$. Let us take two meromorphic functions $f_1\colon Q \to \mathbb{C}$ and $f_2\colon Q \to \mathbb{C}$ given by

$$f_1 = \phi_1/\phi_3 = \phi_4/\phi_2$$

and

$$f_2 = \phi_1/\phi_4 = \phi_3/\phi_2.$$

We can extend them to rational maps from Q to $\mathbb{P}^1$ and take the graphs of these maps. Let us denote them by X_1 and X_2, respectively. In this way we get two small resolutions $\pi_1: X_1 \to Q$ and $\pi_2: X_2 \to Q$ of the singularity of Q at the vertex $V = [0,0,0,0,1]$.

Let Z be obtained from Q by blowing up V in the ordinary way into a smooth 2-dimensional quadric E. The 3-fold Z is a $\mathbb{P}^1$-bundle over $E = \mathbb{P}^1 \times \mathbb{P}^1$. The projection is given by (f_1, f_2). The bundle Z has structural group $\mathbb{C}^*$ with two sections, namely E and the smooth 2-dimensional quadric at ∞ given by $\phi_1\phi_2 - \phi_3\phi_4 = \phi_5 = 0$. We have a commutative diagram:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & X_2 \arrow[d, "\pi_2"] \\
X_1 \arrow[r, "\pi_1"] & Q
\end{tikzcd}$$

In this situation we say that X_1 and X_2 are related by *the Atiyah flop.*

Now let us define *classical flops.* This can be done for algebraic manifolds. Let $\mathcal{X}$ be an n-fold which is singular at a smooth $(n-3)$-fold M. Assume that at each point of M the variety $\mathcal{X}$ is locally Zariski isomorphic to $Q \times M$. Then similarly as above $\mathcal{X}$ has two small resolutions of the singularities at M (fibrewise). Let us call them $\mathcal{X}_1$ and $\mathcal{X}_2$. Then we say that $\mathcal{X}_1$ and $\mathcal{X}_2$ are related by *a classical flop.* Totaro gave the following characterization of genera invariant under classical flops:

3.2. THEOREM [T]. *The following conditions are equivalent:*
(1) a genus φ is elliptic, i.e., it is a specialization of φ_{ell},
(2) φ is invariant under classical flops.

Sketch of the proof. (1)⇒(2) Totaro showed that if $\mathcal{X}_1$ and $\mathcal{X}_2$ are related by the classical flop then $\mathcal{X}_2 - \mathcal{X}_1 = \tilde{\mathbb{P}}^3$-bundle over M in the complex cobordism ring. Therefore, $\varphi(\mathcal{X}_1) - \varphi(\mathcal{X}_2) = \varphi(\tilde{\mathbb{P}}^3) \cdot \varphi(M) = 0$. For (2)⇒(1) it is easy to construct a classical flop between some varieties $\mathcal{X}_1$ and $\mathcal{X}_2$ such that their difference in $\Omega_*^U \otimes \mathbb{Q}$ is equal to a $\tilde{\mathbb{P}}^3$-bundle $X_n \to \mathbb{P}^{n-3}$, where $n \geq 5$ and X_n comes from the chosen base sequence. Therefore $\varphi(X_n) = \varphi(\mathcal{X}_1) - \varphi(\mathcal{X}_2) = 0$ and the quotient map

$$\Omega_*^U \otimes \mathbb{Q} \to \Omega_*^U \otimes \mathbb{Q}/I = \mathbb{Q}[X_1, X_2, X_3, X_4],$$

where I is the ideal generated by differences $\mathcal{X}_1 - \mathcal{X}_2$, is the Krichever–Höhn elliptic genus, Q.E.D.

4. Fundamental differential equation for elliptic genera

Every power series $Q(x) = 1 + a_1x + a_2x^2 + \cdots = xF(x)$ can be written in the form $\frac{x}{1-e^{-x}}P(e^x - 1)$, where $P(y) = 1 + b_1y + b_2y^2 + \ldots$ is some other power series. Hence every genus for compact complex manifolds X can be written as a linear combination of expressions $\chi(X, W_j)$, where W_j are bundles associated to the tangent bundle by representations.

4.1. Let X be a compact n-dimensional complex manifold and let S^1 act on X by holomorphic maps. If we have a genus φ belonging to a power series Q, then for $\lambda \in S^1$ the equivariant genus $\varphi(X, \lambda)$ is defined by the equivariant holomorphic Euler numbers $\chi(X, W_j, \lambda)$ defined as alternating sum of the traces of the action in the cohomology groups $H^i(X, \mathcal{O}(W_j))$. In fact, the holomorphic Lefschetz theorem

(4.2) shows that $\varphi(X,\lambda)$ does not depend on how we write the genus in terms of holomorphic Euler numbers of associated bundles.

4.2. The fixed point locus X^{S^1} is a disjoint union of smooth connected submanifolds of X (possibly of different dimensions). To each connected component Y of X^{S^1} one can associate *the rotation numbers* $(r_1,...,r_n)$ such that for each point $x \in Y$ the group S^1 acts on the tangent space T_xX by the diagonal matrix with entries $(\lambda^{r_1},...,\lambda^{r_n})$. Each subset $\{r_i : r_i = k\} \subset \{r_1,...,r_n\}$ leads to the eigenspace bundle $E_k \subset TX|_Y$ on Y (clearly $E_0 = TY$). To each number r_i one can associate the formal root x_i of the total Chern class of the corresponding eigenspace bundle over Y. Therefore for each component Y of the fixed point locus one can define the function:

$$\varphi(X,\lambda)_Y = \prod_{\{j:r_j=0\}} x_j \cdot \prod_{j=1}^{n} F(x_j + 2\pi i r_j z)[Y], \tag{4.2.1}$$

where $\lambda = e^{2\pi i z}$, $z \in \mathbb{R}$.

Then the holomorphic Lefschetz theorem (the Atiyah–Bott–Singer fixed point theorem: [AS, p. 566]) says that

$$\varphi(X,\lambda) = \sum_Y \varphi(X,\lambda)_Y, \tag{4.2.2}$$

where the summation takes place over all connected components Y of the fixed point locus X^{S^1}.

We put $2\pi i z = x$. Then the expression in (4.2.1) is a Laurent series in x with finite principal part. However, the equivariant genus (4.2.2) is a power series in x. Formula (4.2.2) can be used as definition of the equivariant genus, also in the stable almost complex case. For equivariant genera see [K], section 2. Let x be the first Chern class of the universal S^1-bundle with X as fibre. Then $\varphi(X,\lambda)$ gives the integration over the fibre of the total genus of the bundle along the fibres.

If Y is an isolated fixed point, then

$$\varphi(X,\lambda)_Y = F(r_1x)F(r_2x)\dots F(r_nx).$$

If Y has codimension 1 with normal rotation number 1, then $\varphi(X,\lambda)_Y$ is a linear combination of $F(x)$, $F'(x)$,..., $F^{(n-1)}(x)$.

Proof:

$$\varphi(X,\lambda)_Y = Q(x_1)\dots Q(x_{n-1})F(x_n + x)[Y]$$

But $F(x_n + x) = \sum_{j=0}^{n-1} \frac{F^{(j)}(x)}{j!} x_n^j$.

4.3. Introduce the following S^1-action on the quadric Q (see 3.1):

$$(\phi_1,\phi_2,\phi_3,\phi_4,\phi_5) \to (\lambda\phi_1,\phi_2,\lambda\phi_3,\phi_4,\phi_5). \tag{4.3.1}$$

This action is compatible with the Atiyah flop. We have $(f_1,f_2) \to (f_1,\lambda f_2)$. We can naturally lift it to X_1, X_2 and Z.

Let us consider the fixed point locus for this action on Q. There are two connected components: the plane $\Pi \simeq \mathbb{P}^2$ given by $\phi_1 = \phi_3 = 0$ and the line given by $\phi_2 = \phi_4 = \phi_5 = 0$ in the ∞ hyperplane. The maps π_1 and π_2 are isomorphisms off the vertex V and hence the only interesting components of the fixed point locus on X_1 and X_2 lie over the plane Π.

Namely, for the calculation of $\varphi(X_1,\lambda)-\varphi(X_2,\lambda)$ for some genus φ the fixed component in the ∞ hyperplane cancels out. In X_1 the preimage $\pi_1^{-1}(\Pi)$ is Π blown up at V because $f_1|_\Pi$ is indeterminate at V. This component of the fixed point locus will be denoted by Σ_1. In X_2 we have

$$\pi_2^{-1}(\Pi)=\Pi'\cup\pi_2^{-1}(V)$$

where Π' is a plane intersecting the exceptional line $\pi_2^{-1}(V)$ in the point $f_2=0$. The plane Π' is fixed. We have an isolated fixed point P on $\pi_2^{-1}(V)$ given by $f_2=\infty$. In Σ_1 and Π' the rotation numbers are $(0,0,1)$. In P they are $(-1,1,1)$. Therefore by 4.2 for an arbitrary power series $Q(x)=xF(x)=1+a_1x+a_2x^2+\dots$ and the corresponding genus φ we have

$$\varphi(X_2,\lambda)-\varphi(X_1,\lambda)=F(-x)F(x)F(x)+\frac{F''(x)}{2}+\gamma_1F'(x)+\gamma_2F(x),$$

where $\gamma_1=-a_1$ and $\gamma_2=3a_2-a_1^2$, because the principal parts have to cancel out.

PROPOSITION 4.3.1. *If a genus φ determined by a power series $Q(x)=xF(x)$ is invariant under flops then F satisfies the following differential (functional) equation:*

$$F(x)F(x)F(-x)+\frac{F''(x)}{2}+\gamma_1F'(x)+\gamma_2F(x)=0. \tag{4.3.2}$$

Indeed (4.3.2) is equivalent to $\varphi(X_1,\lambda)=\varphi(X_2,\lambda)$ and this is equivalent to $\varphi(E_1)=\varphi(E_2)$ for all bundles E_1, E_2 with fibres X_1, X_2 associated to an arbitrary $\mathbb{C}^*$-bundle over an arbitrary base B by the action (4.3.1).

In (4.3.2) the coefficients a_i of $Q(x)$ are recursively determined if a_1, a_2, a_3, a_4 are given.

Equation (4.3.2) also follows from a simple action on $\tilde{\mathbb{P}}_3$.

For $(\lambda,\mu)\in S^1\times S^1$ one can study more generally the action

$$(\phi_1,\phi_2,\phi_3,\phi_4,\phi_5)\to(\lambda\mu\phi_1,\phi_2,\lambda\phi_3,\mu\phi_4,\phi_5).$$

The equality

$$\varphi(X_1,\lambda,\mu)=\varphi(X_2,\lambda,\mu)$$

is equivalent to the functional equation

$$F(x+y)F(x)F(-x)-F(x+y)F(y)F(-y)=F'(x)F(y)-F'(y)F(x). \tag{4.3.3}$$

This follows from the fact that the action has a fixed point and a fixed line in both X_1 and X_2 (and three fixed points at ∞ away from V). Compare (4.3.3) with formula (1.9) in [K]. Also (4.3.3) characterizes the characteristic power series of the universal elliptic genus.

5. Elliptic genera and elliptic functions

5.1. From Theorem 2.2.1 and the results of Totaro in Section 3 we know that

$$F(x)=e^{-rx}F_\tau(x,v)$$

is the general solution of (4.3.2). We want to check this directly.

The function $F_\tau(x,v)$ appeared in [Z] in connection with periods of modular forms and it was thoroughly examined there (see Section 3, [Z]). It is symmetric in x and v and (as a function of x) it has poles of order 1 exactly at the points of the lattice $L = 2\pi i(\mathbb{Z}\tau + \mathbb{Z})$. Moreover, it is easy to see that it is almost elliptic with respect to L viewed as depending on the parameter v:

$$F_\tau(x, v + 2\pi i(m\tau + r)) = e^{-mx} F_\tau(x, v) \tag{5.1.1}$$

for $m, r \in \mathbb{Z}$. The same almost ellipticity holds for the variable x. It follows by comparing zeros and poles that

$$F(x)F(-x) = -\wp_L(x) + \wp_L(v) \tag{5.1.2}$$

where $\wp_L$ is the Weierstrass $\wp$-function with respect to the lattice L. The preceding formula can also be deduced from classical formulas concerning the σ-function (1.5.5). If we take the function $F(x)$ with $a_1 = 0$, then all coefficients a_k are elliptic functions in v with respect to L, and (5.1.2) shows that $2a_2 = -\wp_L(v)$. Equation (4.3.2) is invariant if one multiplies $F(x)$ with e^{sx}. Therefore, we can assume $\gamma_1 = -a_1 = 0$ and $\gamma_2 = 3a_2 = -\frac{3}{2}\wp_L(v)$. In view of (5.1.2) the equation (4.3.2) becomes

$$F''(x) - 2\wp_L(x)F(x) = \wp_L(v)F(x), \tag{5.1.3}$$

which is the Lamé differential equation ([K], (1.7)), establishing $F(x)$ as eigenfunction for the differential operator $\frac{\partial^2}{\partial x^2} - 2\wp_L(x)$.

5.2. It follows from (5.1.1) that the genus φ_{ell} is an elliptic function for the lattice L if the first Chern class of the manifold vanishes. It is also independent of r. Therefore we use $F(x) = F_\tau(x, v) = \frac{1}{x} + \frac{1}{v} + \dots$. For a manifold X of dimension n with first Chern class 0 the elliptic function $\varphi_{ell}(X)$ has a principal part beginning with $\frac{c_n(X)}{v^n}$. We have the following table:

	φ_{ell}
$\mathbb{P}^1$	A
K3-surface	$B = 24\wp_L(v)$
S^6	$C = -\wp'_L(v)$
$\tilde{Q}^4$	$D = \wp''_L(v)$

where A is an arbitrary parameter because in (1.5.4) the parameter r is arbitrary.

In [Z], section 3, we find a formula which gives the modular behaviour of $F_\tau(x, v)$:

$$F_{\frac{a\tau+b}{c\tau+d}}\left(\frac{x}{c\tau+d}, \frac{v}{c\tau+d}\right) = (c\tau+d)e^{\frac{cxv/2\pi i}{c\tau+d}} F_\tau(x, v) \tag{5.2.1}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

We want to specialize (5.2.1) for the case that the characteristic power series of the genus is $xF_\tau(x, \frac{2\pi i}{N})$. We have

$$F_{\frac{a\tau+b}{c\tau+d}}\left(\frac{x}{c\tau+d}, \frac{2\pi i}{N}\right) = (c\tau+d)e^{\frac{cx}{N}} F_\tau(x, \frac{2\pi i}{N}(c\tau+d)).$$

If $c \equiv 0\,(N)$ and $d \equiv 1\,(N)$, then by (5.1.1)

$$F_{\frac{a\tau+b}{c\tau+d}}\left(\frac{x}{c\tau+d}, \frac{2\pi i}{N}\right) = (c\tau+d)F_\tau(x, \frac{2\pi i}{N}).$$

If we consider the genus for the power series

$$Q_\tau(x) = xF_\tau(x, \frac{2\pi i}{N})$$

we have for $c \equiv 0\,(N)$ and $d \equiv 1\,(N)$ (i.e., $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$)

$$Q_{\frac{a\tau+b}{c\tau+d}}\left(\frac{x}{c\tau+d}\right) = Q_\tau(x)$$

and hence for a stable almost complex manifold of dimension n the genus is a modular form for the group $\Gamma_1(N)$ (These genera were studied in [HBJ]). In particular A, B, C, D are such modular forms where $\frac{A}{2} = \zeta(\frac{2\pi i}{N}) - \frac{\eta_1}{N}$. For $N = 2$, the values of A and C are 0. We have the genus mentioned in 2.1.

5.3. If the first Chern class of X vanishes, then we get the modular property of φ_{ell} for the full group of $SL_2(\mathbb{Z})$. The formula (5.2.1) implies

$$\varphi_{ell, \frac{a\tau+b}{c\tau+d}}\left(X, \frac{v}{c\tau+d}\right) = (c\tau+d)^n \varphi_{ell,\tau}(X, v). \tag{5.3.1}$$

Here $\dim X = n$. The dependence of φ_{ell} on τ and v is indicated in the notation of (5.3.1). We have seen that $\varphi_{ell}(X) \in \mathbb{C}[\wp, \wp', \wp'']$ and therefore (5.3.1) is clear anyhow.

5.4. The Weierstrass $\wp$-function satisfies the equations

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 \tag{5.4.1}$$

and

$$2\wp'' = 12\wp^2 - g_2. \tag{5.4.2}$$

The discriminant $\Delta = g_2^3 - 27g_3^2$ can be written in terms of B, C and D as follows ([Hö])

$$\Delta = -\frac{1}{32}B^3C^2 + \frac{9}{2}BC^2D + \frac{1}{16}B^2D^2 - 27C^4 - 8D^3.$$

The χ_y-genus applied to SU-manifolds ($c_1 = 0$) has as image $\mathbb{C}[B, C, D]/\Delta$ (see [Hö], [T]). Up to dimension 11 the genera φ_{ell} and χ_y give the same information. The normalized χ_y-genus is the specialization of φ_{ell} for $q = 0$ (and $y = -e^{-v}$). The values of χ_y for B, C, D in the table in 2.2 give a parameterization of $\Delta = 0$.

Example: For the Euler number $g_2 = g_3 = 0$ and for the signature $g_2 = 4/3$, $g_3 = -8/27$. In both cases $\Delta = 0$.

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