F. H I R Z E B R U C H

ARITHMETIC GENERA AND THE THEOREM OF RIEMANN-ROCH

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INTRODUCTORY LECTURE.

I am very glad that I can give a series of lectures in this International Mathematical Summer Seminar at the Lake of Como. It is a great honor for me, and I wish to thank you very much for your kind invitation.

The purpose of these lectures is to show how the theorem of Riemann-Roch can be formulated and proved for non singular algebraic varieties of arbitrary dimension.

I am in the process of writing a report for the "Ergebnisse der Mathematik" (Springer Verlag). This report will contain an introduction to the theory of sheaves (Leray, H. Cartan), the theory of characteristic cohomology classes, the theory of "cobordisme" (R. Thom), and to recent work of A. Borel, K. Kodaira, J.P. Serre, and D.C. Spencer. The report will then contain a detailed discussion and proof of the theorem of Riemann-Roch using all the theories just mentioned.

In these lectures it is impossible to give complete proofs. I have to refer to my report. The lectures will, however, run somewhat along the lines of my report. In this introductory lecture I give a brief account of the whole story (compare the introduction of my "Ergebnisse-report").

1. By an algebraic variety V_n we mean always a compact complex manifold of complex dimension n (not necessarily connected) which can be imbedded complex analytically and without singularities in a complex projective space of sufficiently high dimension.

Let us recall four definitions for the arithmetic genus of an n-dimensionsl algebraic variety V_n. Using the postulation formula (Hilbert's characteristic funetion) we define the integers

$p_a(V_n)$ and $P_a(V_n)$. We call this the <u>1.and 2.definition</u>. <u>Severi</u> conjectured the following formula (see, for example, [1])

$$(1) \qquad p_a(V_n) = P_a(V_n) = g_n - g_{n-1} + \dots + (-1)^{n-1} g_1 \quad ,$$

were g_i denotes the number of complex linearly independent holomorphic differential forms of degree i on V_n. Using the theory of sheaves, <u>Kodaira</u> and <u>Spencer</u> [2] were able to give a simple proof of (1). The alternating sum of the g_i is the <u>3. definition</u> of the arithmetic genus of V_n, thus the equation (1) states that definitions 1,2,3 coincide. The ordering of the g_i in the alternating sum is unnatural. Changing the classical notation we define

$$(2) \qquad \chi(V_n) = \sum_{i=0}^{n} (-1)^i g_i$$

<u>We call $\chi(V_n)$ the arithmetic genus of the algebraic variety</u> V_n. The integer g_0 equals the number of connectedness components of V_n. Thus for a connected variety the classical arithmetic genera p_a, P_a are related to the arithmetic genus χ by the formula

$$1+(-1)^n p_a(V_n) = 1+(-1)^n P_a(V_n) = \chi(V_n).$$

The arithmetic genus χ behaves multiplicatively, i.e., the genus of the cartesian product of two varieties is equal to the product of the genera of the factors. The arithmetic genus in the old definition obviously does not have this multiplicative property.

2. The <u>4.definition</u> of the arithmetic genus is due to <u>J.A.Todd</u> [3] , who showed that the arithmetic genus can be expressed by means of the canonical classes of <u>Eger-Todd</u> [4] . For the dimension 2 and 3 this fact had been established earlier by

M.Noether, Severi and B.Segre. Todd's proof is not complete. It is based on a lemma of Severi for which a precise proof does not seem to exist in the literature.

The Eger-Todd class k_i of V_n is defined as a class of algebraic cycles of V_n of real dimension 2n-2i with respect to an equivalence relation which implies the equivalence relation "homologous" without being identical with it. K_1, for example, is the class of the canonical divisors of V_n with respect to linear equivalence. The class K_i defines a(2n-2i)-dimensional homology class which corresponds under the natural isomorphism to a 2i-dimension cohomology class (with integral coefficients). Up to the sign $(-1)^i$ this cohomology class coincides with the Chern class c_i of the tangent bundle of V_n. We only have to use the Chern classes. Namely, we define the Todd genus $T(V_n)$ directly by means of the Chern classes. It is a principal theorem that $\chi(V_n)$ and $T(V_n)$ are identical for all algebraic varieties.

3. We now come to the definition of $T(V_n)$: In a purely algebraic way we define a polynomial T_n of weight n in indeterminates $c_1,\dots,c_n$ and with rational coefficients.

$$T_o = 1$$

$$T_1 = \frac{c_1}{2}$$

$$T_2 = \frac{1}{12}(c_1^2 + c_2)$$

(3) $$T_3 = \frac{1}{24} c_1 c_2$$

$$T_4 = \frac{1}{720}(-c_4+c_3c_1+3c_2^2+4c_2c_1^2-c_1^4)$$

$$\vdots$$

If we interpret the indeterminate c_i as the 2i-dimensional

Chern class of V_n, and, moreover, the product in the polynomial T_n as the cup product of the cohomology ring of V_n, then T_n is a 2n-dimensional cohomology class of V_n. By the fundamental cycle of V_n we mean that element of the homology group $H_{2n}(V_n,Z)$ which is defined by the natural orientation of V_n. The value taken by the class T_n on the 2n-dimensional fundamental cycle of V_n is a rational number which by definition is the Todd genus $T(V_n)$.

The polynomials T_n should satisfy two properties. First, they should be of such a kind that the functional $T(V_n)$ defined by them behaves multiplicatively as the arithmetic genus does. The sequences of polynomials fulfilling this multiplicative property are called multiplicative sequences. They are characterised by purely formal algebraic conditions. Secondly, the sequence T_n of polynomials should be such that $T(P_n)$ equals 1 for all complex projective spaces P_n. (Observe that $\chi(P_n)=1$ for all n.) This second condition is also a formal condition for the polynomials. By these two conditions our polynomials T_n are characterised.

4. The divisors of an algebraic variety V_n constitute an abelian group which we write additively. Besides the divisors we have complex analytic line bundles over V_n (with fibre C and structural group C^*).

Here C denotes the field of complex numbers, and C^* is the multiplicative group of non-vanishing complex numbers acting on C by multiplication. If we regard isomorphic complex line bundles as identical, the complex line bundles over V_n constitute an abelian group which we write multiplicatively. The group operation is the tensor product. Each divisor defines a complex line bundle. Two divisors define the same complex line bundle, if and only if they are linearly equivalent. In this way we obtain an isomorphism of the group of divisor classes into the group of complex line bundles, under which the sum of divisors goes over into the tensor product of the corresponding line bundles. It was shown by Kodaira and Spencer [5] that for an algebraic variety V_n this isomorphism is into.

Let D be a divisor of V_n, let $H^0(V_n,D)$ be the complex vector space of all those meromorphic functions f on V_n whose divisors (f) added to D give a divisor (f)+D which has no poles. This vector space has always a finite dimension over C. The problem of Riemann-Roch is to determine this dimension.

Now, let F be the complex line bundle corresponding to D. It can be readily shown that the vector space $H^0(V_n,D)$ is isomorphic to the vector space $H^0(V_n,F)$ of all holomorphic sections of F. (Observe that $\dim H^0(V_n,D)$ depends only on the divisor class of D).

5. We have pointed out already that one of the principal theorems is the equation

$$\chi(V_n)=\sum_{i=0}^{n}(-1)^i g_i = T(V_n). \tag{4}$$

The value of the Chern class c_n of V_n on the fundamental cycle of V_n equals the Euler number of V_n. Hence (4) gives for connected algebraic curves

$$(4)_1 \qquad \chi(V_1)=1-g_1=\frac{1}{2}(2-2p)$$

Here p denotes the number of handles of V_1. The theorem of Riemann-Roch states for algebraic curves

$$(4)_1^* \qquad \dim H^0(V_1,D)-\dim H^0(V_1,K-D)=d+1-p,$$

where d is the degree of the divisor D and where K is a canonical divisor. The equation $(4)_1$ can be obtained from $(4)_1^*$ by putting

D = 0. We shall show that (4) admits a generalisation $(4)^*$ analogously to the generalisation from $(4)_1$ to $(4)_1^*$. Let us now use complex line bundles instead of divisor classes. We know already that this is essentially the same.

Let $H^i(V_n,F)$ be the i-dimensional cohomology group of V_n with coefficients in the sheaf of germs of holomorphic sections of the line bundle F. If F is the product bundle 1 (trivial line bundle) which corresponds to the zero divisor then we have the sheaf of local holomorphic functions. The "group" $H^i(V_n,F)$ is a vector space over C. The group $H^0(V_n,F)$ is the vector space we have discussed in 4., when formulating the problem of Riemann-Roch. According to Dolbeault [6] we have dim $H^i(V_n,1)=g_i$. Thus the natural generalisation of g_i will be the number dim $H^i(V_n,F)$. According to a theorem of Cartan-Serre [7] and Kodaira [8], the vector space $H^i(V_n,F)$ is finite dimensional. Therefore the number dim $H^i(V_n,F)$ is actually defined. It vanishes for $i > n$. Now we introduce the Euler-Poincaré characteristic

$$(5) \qquad \chi(V_n,F) = \sum_{i=0}^{n} (-1)^i \dim H^i(V_n,F).$$

This is the generalisation of the left side of (4). We shall show that $\chi(V_n,F)$ can be expressed as a polynomial in the cohomology class f of the line bundle F and the Chern classes c_i of V_n. Before writing down these polynomials let us recall that the cohomology class f of F is a two dimensional integral cohomology class of V_n which can be defined as the first Chern class of F, i.e., as the first obstruction for a non-vanishing continuous section of F. If F is given by a divisor D, then f is that cohomology class of V_n which corresponds to the (2n-2)-dimensional integral homology class represented by the cycle D.

In the following polynomials the product has again to be interpreted as cup-product. If b is a 2n-dimensional cohomology class, then $b\,[V_n]$ denotes the value of b on the fundamental cycle of V_n. Now let us write down the polynomials for $\chi(V_n,F)$.

$$\chi(V_1,F) = (f + \frac{c_1}{2})\,[V_1]$$

$$(4)^{*} \qquad \chi(V_2,F) = (\frac{f^2}{2} + \frac{fc_1}{2} + \frac{1}{12}(c_1^2 + c_2))\,[V_2]$$

$$\chi(V_3,F) = (\frac{1}{6} f^3 + \frac{1}{4} f^2 c_1 + \frac{1}{12} f(c_1^2 + c_2) + \frac{1}{24} c_1 c_2)\,[V_3].$$

$$\chi(V_n,F) = (\sum_{k=0}^{n} \frac{f^{n-k}}{(n-k)!} T_k(c_1,\ldots,c_k))\,[V_n]$$

This is the generalisation of the theorem of Riemann-Roch to algebraic varieties of arbitrary dimensions.

According to a duality theorem of Serre [9], we have

$$\dim H^k(V_n,F) = \dim H^{n-k}(V_n,KF^{-1}).$$

Here K denotes the canonical line bundle corresponding to the canonical divisor class. Using this duality relation the equation for $\chi(V_1,F)$ and $\chi(V_2,F)$ goes over into the classical theorem of Riemann-Roch for n=1 and n=2. In the case n=2, the number $\dim H^1(V_2,F)$ equals the superabundance of F, i.e., the superabundance of the divisor class corresponding to F.

Kodaira [10] and Serre [11] have sufficient conditions under which the cohomology groups $H^i(V_n,F)$ all vanish for $i > 0$. If these conditions are fulfilled, then $\chi(V_n,F) = \dim H^0(V_n,F)$ and our formula for $\chi(V_n,F)$ solves the problem of Riemann-Roch.

For n=1 the Kodaira-Serre conditions are by virtue of the duality theorem nothing else but the well-known fact that in $(4)_1^{*}$ the term $\dim H^0(V_1,K-D)$ vanishes, if $d > 2p-2$.

6. We now come to a further generalisation of (4). Let W be a complex analytic vector space bundle of V_n (fibre: C_q = vector space of dimension q over the field of complex numbers, group: the general linear group GL(q,C) of all non singular q by q complex matrices).

We define

$$(6) \qquad \chi(V_n, W) = \sum_{i=0}^{n} (-1)^i \dim H^i(V_n, W) ,$$

where $H^i(V_n,W)$ denotes the i-dimensional cohomology group of V_n with coefficients in the sheaf of germs of local holomorphic sections of W. The groups $H^i(V_n,W)$ are again finite dimensional vector spaces over C which vanish for $i > n$. We shall see that $\chi(V_n,W)$ can be expressed as polynomials in the Chern classes c_i of V_n, i.e., the Chern classes of the tangent bundle of V_n, and in the Chern classes of W.

This result can be applied to special vector space bundles over V_n. Let us take the vector space bundle $T^{(p)}$ of the covariant p vectors. We put

$$(7) \qquad \chi^p(V_n) = \chi(V_n, T^{(p)}) .$$

The Chern classes of $T^{(p)}$ can be expressed as polynomials in the Chern classes c_i of V_n. Therefore we obtain for $\chi(V_n)$ a polynomial of weight n in the c_i. According to Dolbeault [6], we have

$$\dim H^q(V_n, T^{(p)}) = h^{p,q}$$

where $h^{p,q}$ denotes the dimension of the complex vector space of all harmonic forms of type (p,q) on V_n. Thus we have

$$\chi^p(V_n) = \sum_{q=0}^{n} (-1)^q h^{p,q} .$$

For p = 0 we get

$$\chi^0(V_n) = \chi(V_n) = \sum_{q=0}^{n} (-1)^q h^{0,q} = \sum_{i=0}^{n} (-1)^i g_i .$$

As an example we give the formula for $\chi^1(V_4)$

(8) $$\chi^1(V_4) - 4\chi(V_4) - \frac{1}{12}\left(2c_4 + c_3 c_1\right)[V_4]$$

For $\chi(V_4)$ see (3).

We see immediately that the sum $\sum_{p=0}^{n} \chi^p(V_n)$ vanishes, if n is odd. For n even Hodge [12] proved that $\sum_{p=0} \chi^p(V_n)$ equals the index of V_n which will be denoted by $\mathcal{J}(V_n)$. This index is a topological invariant of V_n (n even) and is defined as follows. Take the n-dimensional real cohomology group of V_n. This is a vector space $\mathfrak{R}$ over the real field. For an element x of $\mathfrak{R}$ the square x^2 in the sense of the cup product defines the real number $x^2[V_n]$. Since n is even, $x^2[V_n]$ is a quadratic form over $\mathfrak{R}$ which is non-singular. The number of positive eigenvalues minus the number of negative eigenvalues of this quadratic form is the index of V_n.

By the theorem of Hodge and by our polynomials for $\chi^p(V_n)$ we obtain a polynomial for the index. This polynomial is even a polynomial in the Pontrjagin classes of V_n which are defined for arbitrary differentiable manifolds.

7. We have seen that the index of an algebraic V_{2k} is a polynomial in the Pontrjagin classes. Actually, this is the starting point of our considerations. Namely, by the theory of Thom [13] we will be able to prove that the index $\mathcal{J}(M^{4k})$ of a 4k dimensional oriented differentiable manifold M^{4k} can be expressed as a polynomial in the Pontrjagin classes p_i of M^{4k} (p_i is a 4i-dimensional integral cohomology class of M^{4k}). As examples we state

$$\mathcal{J}(M^4) = \frac{1}{3} p_1 [M^4]$$

$$\mathcal{J}(M^8) = \frac{1}{45}\left(7p_2 - p_1^2\right)[M^8]$$

$$\mathcal{J}(M^{12}) = \frac{1}{3^3 \cdot 5 \cdot 7}\left(62p_3 - 13p_2p_1 + 2p_1^3\right)[M^{12}]$$

The formula for $\mathfrak{I}(M^4)$ was conjectured by Wu. The formulas for $\mathfrak{I}(M^4)$ and $\mathfrak{I}(M^8)$ were proved by Thom .

For a short survey of the way from the formula for $\mathfrak{I}(M^{4k})$ to the formula for $\chi(V_n,W)$ we refer to the note [14] .

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