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CHARACTERISTIC CLASSES AND HOMOGENEOUS SPACES, III.*

By A. BOREL and F. HIRZEBRUCH.

This paper consists of three parts, related to each other only by the fact that they bring complements to [1].

In [1, §§ 25, 26], certain expressions (\hat{A} -genus, Chern characters of bundles over spheres, etc.) were proved to be integers “exc 2,” that is, up to a power of two. This restriction came from the fact that the proofs relied heavily on the integrality “exc 2” of the Todd genus of an almost complex manifold proved in [5]. Since then Milnor [8, 12] has shown the Todd genus to be an integer. This fact will be used in § 3 to free our earlier results from the powers of two. For this, it will be necessary to generalize slightly the notion of almost complex manifold, and to introduce between vector bundles an equivalence relation (called here S -equivalence), in which the trivial bundles form one class. These preliminaries are dealt with in §§ 1, 2.

In [1, § 23.3], it was proved that the A -genus of a coset space G/U is zero when G and U are compact, connected, semi-simple, of the same rank. The proof made use of a lemma (23.4) stating that the sum of the positive roots of U is singular in G , which was proved essentially by case by case checking. § 4 brings an a priori proof of this lemma, in the framework of the theory of roots. When all roots of G have the same length, 23.4 is equivalent to a theorem of de Siebenthal [10] saying that the “main diagonal” of U is singular in G . We also give a general proof of this result, which is obtained in [10] by case by case checking.

Finally, § 5 gives two elementary sufficient conditions under which the Stiefel-Whitney class $w(M)$ or the Pontrjagin class $\tilde{p}(M)$ (see [1, § 9.3]) of a compact manifold M reduces to 1, which are then applied to G/T .

The notation of [1] will be used freely.

1. S -classes of vector bundles.

1.1. *Notation.* L stands for the field either of real numbers \mathbf{R} , or of complex numbers \mathbb{C} , or of quaternions \mathbf{K} . $GL(n, L)$ (resp. $U(n, L)$) is the

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general linear group (resp. unitary group) in L^n . A bundle with typical fibre L^n and structural group $\mathbf{GL}(n, L)$ or $\mathbf{U}(n, L)$ is called an L -vector bundle.

1.2. DEFINITION. *Let X be a topological space. Two L -vector bundles ξ, η over X are said to be S -equivalent (suspension equivalent) if there exist trivial bundles α, β such that the Whitney sums $\xi \oplus \alpha$ and $\eta \oplus \beta$ are equivalent bundles in the usual sense. The S -equivalence class, or S -class, of ξ will be denoted by $[\xi]$, and $K'(X, L)$ will be the set of S -classes of L -vector bundles over X .*

Let ξ, η be two L -vector bundles. $[\xi] = [\eta]$ means that the associated principal bundles become equivalent after the standard extension of the structural group to $\mathbf{U}(N, L)$, for some N , or also that the associated unit sphere bundles become equivalent after iterated suspension of the fibres.

The Whitney sum is commutative, associative, and clearly compatible with S -equivalence. Therefore it defines in $K'(X, L)$ a commutative, associative, operation for which the S -class of the trivial bundle is a zero element.

Let $f: X \rightarrow Y$ be a continuous map. Then, if we associate to an L -vector bundle over Y the induced bundle on X , we define clearly a homomorphism of $K'(Y, L)$ into $K'(X, L)$.

1.3. PROPOSITION. *Let X be a locally compact, paracompact, finite dimensional space. Then $K'(X, L)$ is a commutative group (with respect to Whitney sum). The S -class of the trivial bundle is the zero element.*

There remains only to show the existence of the inverse. On the Grassmann manifold $\mathbf{U}(n + N, L)/\mathbf{U}(n, L) \times \mathbf{U}(N, L)$ there are two canonical L -vector bundles ξ, η with typical fibres L^n, L^N , whose sum is the trivial bundle. Hence $[\xi] + [\eta] = 0$. Since, by the classification theorem, any L -vector bundle with typical fibre L^n over X is induced from ξ by a map of X into the Grassmannian (for N suitably large), our assertion follows immediately.

1.4. The total Chern class of a complex vector bundle depends only on its S -class, as follows from the multiplication theorem [1, § 9.7]. It belongs to the set $\Gamma(X, \mathbf{Z})$ of elements of $H^*(X, \mathbf{Z})$ having a zero dimensional term equal to 1, and vanishing odd dimensional components. $\Gamma(X, \mathbf{Z})$ is a commutative group under the cup-product, and it is clear that assigning to each complex vector bundle its Chern class yields a group homomorphism of $K'(X, \mathbf{C})$ into $\Gamma(X, \mathbf{Z})$. An analogous remark can be made of course for the Pontrjagin, symplectic Pontrjagin and Stiefel-Whitney classes.

2. Weakly almost complex structures.

2.1. The standard inclusion of $GL(n, \mathbf{C})$ into $GL(2n, \mathbf{R})$ induces obviously a homomorphism of $K'(X, \mathbf{C})$ into $K'(X, \mathbf{R})$ to be denoted by λ . An S -class of real vector bundles is said to admit (to have) a complex structure if it belongs to the image of λ (and if an element of its inverse image has been chosen). A real vector bundle ξ admits (has) a *weak complex structure* if $[\xi]$ admits (has) a complex structure. Thus, a weak complex structure of a real vector bundle ξ is given by a trivial bundle α and a complex structure of $\xi \oplus \alpha$ in the usual sense [1, § 7.3]. Finally, a manifold is *weakly almost complex* (admits a weak almost complex structure) if its tangent bundle has been endowed with (admits) a weak complex structure.

An orientation of a real vector bundle ξ with fibre \mathbf{R}^q is a section of the associated bundle with $\mathbf{O}(q)/\mathbf{SO}(q)$ as fibre. Since

$$\mathbf{O}(q)/\mathbf{SO}(q) \rightarrow \mathbf{O}(q+1)/\mathbf{SO}(q+1)$$

is bijective, an orientation of ξ depends only on the S -class of ξ . Thus, a weak complex structure of ξ defines an orientation of ξ . In particular, a *weakly almost complex manifold is canonically oriented*.

2.2. *Chern classes.* The Chern class of a weakly almost complex manifold X is by definition the Chern class of the weak complex structure of its tangent bundle. If X is compact, of dimension $2n$, then $c_n[X]$ is not necessarily the Euler number. For instance, take for X the unit sphere in \mathbf{R}^{2n+1} . The normal bundle and its Whitney sum with the tangent bundle are trivial. Therefore S_{2n} admits a weak almost complex structure defined by a trivial complex bundle. Then $c_n[S_{2n}] = 0$.

2.3. *Submanifolds of codimension 2.* Let X be a compact weakly almost complex manifold, ξ its real tangent bundle, and $[\xi']$ the complex structure of $[\xi]$. Let $d \in H^2(X, \mathbf{Z})$. According to Thom [11] there exists an oriented submanifold D of X , of codimension 2, whose normal bundle ν is a real vector bundle with structure group $\mathbf{SO}(2)$ and characteristic class $i^*(d)$, where i is the embedding of D in X . Since $\mathbf{SO}(2) = \mathbf{U}(1)$, the bundle ν has a complex structure ν' , whose total Chern class is $1 + i^*(d)$. Let δ be the real tangent bundle to D . Then we have in $K'(D, \mathbf{R})$ the equalities

$$[\delta] = [i^*\xi] - [\nu] = \lambda(i^*[\xi'] - [\nu']),$$

which show that $[\delta]$ admits a complex structure represented by a bundle δ' whose Chern class is

$$c(\mathcal{S}') = i^*(c(\xi')) \cdot (1 + i^*(d))^{-1}.$$

This proves the following proposition.

2.4. PROPOSITION. *Let X be a compact weakly almost complex manifold and $c(X)$ be its total Chern class. Then every element $d \in H^2(X, \mathbf{Z})$ is representable by a submanifold D of codimension 2 which carries a weakly almost complex structure, whose total Chern class $c(D)$ is equal to $i^*(c(X) \cdot (1 + d)^{-1})$, where i is the embedding of D in X .*

2.5. Let X be a compact weakly almost complex manifold of even dimension, $d \in H^2(X, \mathbf{R})$, and η a complex vector bundle over X . The Todd genus $T(X)$, the virtual Todd genus $T(d)_X$ of d , and the number $T(X, \eta)$ are then defined in exactly the same way as for an almost complex manifold. If η is a complex line bundle, with first Chern class a , then $T(X, \eta) = T(X, a)$, where $T(X, a) = T(X) - T(-a)_X$. For all this, see [5, §§ 10-12]; the definitions given there were also recalled in [1, §§ 22.1, 25.1]. It follows then from 2.4 that for every element $d \in H^2(X, \mathbf{Z})$, the virtual Todd genus $T(d)_X$ is equal to the Todd genus of some compact weakly almost complex manifold.

2.6. Milnor [8] (see also [12]) has established a complex analogue of cobordism theory, and has proved that the Todd genus of a weakly almost complex manifold is an integer. This (and 2.5) yield the

PROPOSITION. *Let X be a compact weakly almost complex manifold. Then for every $d \in H^2(X, \mathbf{Z})$, the number $T(X, d)$ is an integer.*

3. Integrality theorems for differentiable manifolds. For the definition of $\hat{A}(X, d)$ and $\hat{A}(X, d, \eta)$ we refer to [1, §§ 25.4, 25.5].

3.1. THEOREM. *Let X be a compact oriented differentiable manifold and d an element of $H^2(X, \mathbf{Z})$ whose restriction mod 2 is equal to $w_2(X)$. Then $\hat{A}(X, \frac{1}{2}d)$ is an integer.*

We use the notation of 25.4. Thus E/T is an almost complex manifold, $\pi: E/T \rightarrow X$ a fibre map, ξ the tangent bundle to X , and $x_1, \dots, x_q \in H^2(E/T)$ are the roots of the Chern polynomial of a complex structure of $\pi^*(\xi)$. This implies that

$$\pi^*(w_2) = x_1 + \dots + x_q \pmod{2}.$$

Furthermore, we have the equality

$$(1) \quad \hat{A}(X, \frac{1}{2}d) = T(E/T, \frac{1}{2}(\pi^*(d) - (x_1 + \dots + x_q))).$$

If now $d \equiv w_2 \pmod 2$, then $\pi^*(d) - (x_1 + \dots + x_q) \equiv 0 \pmod 2$, therefore the real cohomology class $\frac{1}{2}(\pi^*(d) - (x_1 + \dots + x_q))$ comes from an integral class under the coefficient homomorphism $\mathbf{Z} \rightarrow \mathbf{R}$. Theorem 3.1 follows then from (1) and 2.6.

3.2. COROLLARY. *Let X be a compact, oriented, differentiable manifold with vanishing second Stiefel-Whitney class. Then the genus $\hat{A}(X)$, belonging to the power series $\frac{1}{2}z^{\frac{1}{2}}/\sinh \frac{1}{2}z^{\frac{1}{2}}$, is an integer.*

In this case, we can replace d by 0 in 3.1. Since $\hat{A}(X, 0) = \hat{A}(X)$, the corollary follows.

3.3. Examples. The polynomial \hat{A}_1 is equal to $-p_1/24$. Thus, if $\dim X = 4$,

$$\hat{A}(X, d/2) = ((1 + d/2 + d^2/8)(1 - p_1/24)) [X] = (d^2/8 - p_1/24) [X].$$

Since $p_1[X] = 3 \cdot \tau$, where τ is the index of X (see [5, § 0.7; 11, Cor. IV.13]), we get on a 4-dimensional oriented manifold the congruence

$$d^2[X] \equiv \tau \pmod 8 \quad (d \in H^2(X, \mathbf{Z}), d \equiv w_2 \pmod 2).$$

This can also be formulated as a statement on the quadratic form of the manifold (F. Hirzebruch-H. Hopf, *Mathematische Annalen*, vol. 136 (1958), pp. 156-172).

Let now X be 6-dimensional. Then

$$\hat{A}(X, d/2) = ((1 + d/2 + d^2/8 + d^3/48)(1 - p_1/24)) [X]$$

yields the congruence

$$d^3[X] \equiv (d \cdot p_1) [X] \pmod{48}, \quad (d \in H^2(X, \mathbf{Z}), d \equiv w_2 \pmod 2).$$

3.4. The coefficient of p_k in \hat{A}_k is $-B_k / ((2k!) \cdot 2)$, where B_k is the k -th Bernoulli number [5, p. 13]. This can be easily deduced from [5, § 1]. In fact, by the usual expression of the Pontrjagin classes in terms of Chern classes [5, p. 12], $p_k = (-1)^{k/2} \cdot c_{2k}$, modulo decomposable elements; since $A_k = 2^{k/2} \hat{A}_k$, formula (12) of [5, p. 15] shows that the coefficient of p_k is $(-1)^{k/2}$ times the coefficient of c_{2k} in T_{2k} ; but this coefficient is also the coefficient of c_1^k in T_{2k} [5, Bemerkung 2, p. 15], and it follows readily from the first formula in [5, § 1.7, p. 15] that the latter is equal to $(-1)^{k-1} B_k / (2k)!$, whence our assertion. Together with 3.2, it implies:

3.5. THEOREM. *Let X be a compact oriented differentiable manifold*

of dimension $4k$ whose tangent bundle is trivial when restricted to the complement of some point in X . Then $B_k \cdot p_k[X] / ((2k)! \cdot 2)$ is an integer.

This theorem has found an interesting application to the stable homotopy groups of spheres [7]. (See also [6].)

3.6. THEOREM. *Let X be a compact oriented differentiable manifold, η a complex vector bundle over X , and d an element of $H^2(X, \mathbf{Z})$ whose restriction mod 2 is equal to $w_2(X)$. Then $\hat{A}(X, \frac{1}{2}d, \eta)$ is an integer.*

We follow the notation of [1, §25.5]. Thus σ is the tangent bundle to X , E/T is the total space of a bundle ξ over X with fibre map π , and $a_j \in H^2(E/T, \mathbf{Z})$ ($1 \leq j \leq m$) are cohomology classes whose sum a is the first Chern class of the complex vector bundle along the fibres. Therefore a , reduced mod 2, is the second Stiefel-Whitney class of the bundle along the fibres $\hat{\xi}$. Since the tangent bundle to E/T is the sum of $\pi^*(\sigma)$ and of $\hat{\xi}$, [1, §7.6], we have

$$(2) \quad \pi^*(d) + a \equiv w_2(E/T) \pmod{2}.$$

In [1, §25.5] it is proved that

$$\hat{A}(X, \frac{1}{2}d, \eta) = \sum_i \hat{A}(E/T, \frac{1}{2}(\pi^*(d) + 2x_i + a)), \quad (x_i \in H^2(X, \mathbf{Z})).$$

Therefore, 3.6 follows from 3.1 and (2).

3.7. Applications. Theorem 3.6 gives a positive answer to conjecture (1) in [1, §25.6]. Conjecture (2) ($\hat{A}(X)$ is even if $w_2(X) = 0$ and $\dim X \equiv 4 \pmod{8}$) and also 3.6 have since been proved by a quite different method [6] which uses Bott's results [3] instead of Milnor's theorem. As a consequence, in 3.5, $B_k p_k[X]$ divided by $(2k)! \cdot 2$ is an even integer for odd k , which yields a slight sharpening of the Kervaire-Milnor theorem [7].

In [1, §26.10] we mentioned the *theorem of Bott* [3] that the Chern class of a complex vector bundle over S_{2q} is divisible by $(q-1)!$, which had been proved in [1, §25.8] only "exc 2." This and the corresponding divisibility property of Pontrjagin classes follow now from 3.6, in the same way as [1, §§25.8, 25.9] were derived from [1, §25.5].

Let X be a compact almost complex manifold η a complex vector bundle over X . Then $T(X, \eta) = \hat{A}(X, \frac{1}{2}c_1, \eta)$. Therefore by 3.6, $T(X, \eta)$ is an integer. This gives an affirmative answer to the first question of Problem 22 in [4]. It follows also that all numbers introduced in connection with the Riemann-Roch theorem, which were proved to be integers "exc 2" in [1, §25.6], are actually integers.

4. Some properties of roots of compact Lie groups. G will always be a compact connected semi-simple Lie group, T a maximal torus. No distinction is made between a point in the universal covering V of T and its image in T (or equivalently, between a point in the Lie algebra \mathfrak{t} of T and its image in T under the exponential map); expressions like positive Weyl chamber, dominant root, simple roots are always understood with respect to some ordering. For the notation see [1, § 2].

4.1. If a, b are roots, the number $q(a, b) = 2(a, b)(b, b)^{-1}$ is an integer, and $0 \leq q(a, b) \cdot q(b, a) \leq 3$. Consequences: $q(a, b) = 0, 1, 2, 3$. If $(aa) < (bb)$ and $(ab) \neq 0$, then $q(ab) = \pm 1$. If $q(a, b) = \pm 1$, then $|q(a, b)| \leq |q(b, a)|$, hence $(a, a) \leq (b, b)$; if $(a, b) \neq 0$, and $(a, a) \geq (b, b)$, then $(a, a)(b, b)^{-1} = 1, 2, 3$. Let now G be simple. Then $W(G)$ is irreducible. Since the roots of a given length span a subspace invariant under $W(G)$, it follows that if G has a root of a certain length λ , then for any root a of G , there exists a root of length λ not orthogonal to a . Consequently, if the roots are normalized so that the minimal root length is one, then the other possible values are 2, 3. More precisely, it is known that the values of (a, a) are 1, 3 for $G = \mathbf{G}_2, \mathbf{1}, \mathbf{2}$ for $G = \mathbf{B}_n, \mathbf{C}_n, \mathbf{F}_4$ and 1 in the other cases. We recall that if a, b are roots, then so is $c = a - q(a, b)b$, and clearly $(a, a) = (c, c)$.

4.2. Let G be simple, a_i ($1 \leq i \leq l$) be a system of simple roots, and $d = d_1 a_1 + \dots + d_l a_l$ be the highest root. Then d has maximal length.

For completeness, we give a proof. There exists a positive root of maximal length $c = c_1 a_1 + \dots + c_l a_l$ not orthogonal to d (4.1). If $c = d$, we are done, so assume $d \neq c$. Since d is dominant, $c + d$ is not a root, therefore $(c, d) > 0$, $q(c, d) = k \geq 1$, and $c - kd$ is a root. The coefficient of a_i in $c - kd$ must then be smaller in absolute value than d_i , whence $k = 1$; but then $(c, c) \leq (d, d)$ by 4.1 and d has maximal length.

4.3. Let G be simple, a_i ($1 \leq i \leq l$) be simple roots of G , $d = d_1 a_1 + \dots + d_l a_l$ be the highest root, and U be a maximal connected semisimple subgroup containing T . Then there exists an index j such that d_j is prime, U is the centralizer of the point P_j defined by $d_j \cdot a_i(P_j) = \delta_{ij}$ ($1 \leq i \leq l$). The simple roots of U with respect to a suitable ordering are the a_i 's ($i \neq j$) and $-d$. The roots of U are exactly the roots of G in which a_j has coefficient 0 or $\pm d_j$. They form a closed system (i.e. if a, b are 2 roots of U such that $a + b$ is a root of G , then $a + b$ is a root of U). If the center of G reduces to the identity, then P_j generates a cyclic group of order d_j which is the center

of U . One obtains in this way all maximal connected semi-simple subgroups of maximal rank, up to inner automorphisms. For all this, see [2].

4.4. G being again semi-simple, let a_i ($1 \leq i \leq l$) be its simple roots. Then the equations $a_1 = \cdots = a_l$ define a 1-dimensional subspace contained in the positive Weyl chamber, or also a 1-dimensional torus S in T , to be called the *main diagonal*. It belongs to a three dimensional simple group H , the *principal subgroup* of G in the sense of de Siebenthal, which is defined up to inner automorphisms by those conditions [10, § 13, Th. 2]. H is not contained in a proper subgroup of rank l [10, § 12, Th. 1].

4.5. Let $G = \mathbf{SU}(2)$, and Γ_n be the natural representation of degree n of G in the space of homogeneous forms of degree $n-1$ in two variables. As is known, Γ_n is up to equivalence, the only irreducible representation of degree n of G . If n is odd, it is equivalent to a real representation and not faithful. For n even, Γ_n is faithful, equivalent to the complex conjugate representation but not to a real representation. This implies in particular: if Γ is a real representation of G whose restriction to a maximal torus does not contain the trivial representation, then Γ is faithful, breaks up in a sum of real irreducible representations each of which is complex equivalent to a sum $\Gamma_n + \bar{\Gamma}_n$, n even, hence $\Gamma = \Delta + \bar{\Delta}$, where Δ is a sum of representations Γ_n , n even.

4.6. THEOREM. *Let U be a proper connected semi-simple subgroup of G of maximal rank. Then the main diagonal of U is singular in G [10, § 8, Théorème 7].*

We may assume that the center of G is reduced to the identity. If G is a direct product $G_1 \times G_2$, then $U = U_1 \times U_2$, where $U_i = U \cap G_i$ is a subgroup of maximal rank of G_i (see e.g. [2]), and its main diagonal clearly projects onto the main diagonal of U_i ($i=1, 2$). Using this and induction, the proof of 4.6 is easily reduced to the case where G is simple, with center reduced to (e) , and U is maximal connected. Assuming this from now on, we follow the notation of 4.3 admitting moreover the simple roots to be numbered in such a way that $j=1$. Let c^* be the point of S defined by $a_i(c^*) = 1$ ($2 \leq i \leq l$) and $d(c^*) = -1$. Then

$$(1) \quad d_1 a_1(c^*) = -1 - d_2 - \cdots - d_l.$$

$a(c^*)$ is integral for all roots a of a system of simple roots of U , hence for all roots of U ; therefore, c^* is an element of the center of U .

Assume now that, contrary to our assertion, S is regular. Then $\text{Ad}_{\mathfrak{g}/\mathfrak{u}}S$ does not contain the trivial representation. Let H be the principal subgroup of U containing S . By 4.5, $H = \mathbf{SU}(2)$, $\text{Ad}_{\mathfrak{g}/\mathfrak{u}}H$ is faithful, and is a sum of two equivalent representations. From this, and from standard facts about representations of the circle group, it follows that given a complementary root a , there exists a positive complementary root $b \neq \pm a$ such that $b(s) = \pm a(s)$ for all $s \in S$. In particular, taking $a = a_1$, there exists a positive root $b = b_1a_1 + \dots + b_l a_l$ not proportional to a_1 , such that

$$(2) \quad b_1 a_1(c^*) + b_2 + \dots + b_l = \pm a_1(c^*)$$

($b_i \geq 0$, ($i \geq 1$), $b_1 \geq 1$, $(b_2, \dots, b_l) \neq (0, \dots, 0)$). Let z be the element $\neq e$ in the center of H . Its connected centralizer U_1 in G is $\neq G$, since G has no center, and contains H, T ; the last assertion of 4.4 shows then that U_1 contains U , hence is equal to U , since the latter is maximal connected. Thus, by 4.3, $d_1 = 2$ and $b_1 = 1$. It follows that in the right hand side of (2) we must have the minus sign, and we get

$$(3) \quad -2a_1(c^*) = b_2 + \dots + b_l,$$

but this, together with $b_i \leq d_i$, obviously contradicts (1). Therefore S is singular.

4.7. There exists therefore a positive complementary root $b = b_1a_1 + \dots + b_l a_l$ such that $b(s) = 0$ for all $s \in S$. In particular, $b_1 a_1(c^*) = -(b_2 + \dots + b_l)$ is integral; since $0 < b_1 < d_1$ and d_1 is prime, this and (1) show that $a_1(c^*)$ is integral. We have proved:

COROLLARY. *We keep the notation of 4.3 and assume d_j to be prime. Then $1 + d_1 + \dots + d_l$ is divisible by d_j . If c is the linear form defined by $(a_i, c) = 1$ ($i \neq j$), $(d, c) = -1$, then (a, c) is integral for all the roots a of G .*

Before stating our next theorem, we discuss some more properties of roots.

4.8. *Let G be simple, a_i ($1 \leq i \leq l$) be the simple roots, and $c = c_1 a_1 + \dots + c_l a_l$ be a root of G . Then $c_i(a_i, a_i) \cdot (c, c)^{-1}$ is an integer ($1 \leq i \leq l$).*

It is known that if we perform an inversion with respect to a sphere of radius $2^{\frac{1}{2}}$ in V , then a system of roots is transformed into a system of roots (of a group G' which may or may not be isomorphic to G). Let $e \rightarrow \bar{e}$ be this transformation. Then $\bar{e} = 2e \cdot (e, e)^{-1}$, and in particular

$$\begin{aligned} \bar{c} &= 2 \cdot (c, c)^{-1} \sum_i c_i(a_i, a_i) \cdot 2^{-1} \cdot \bar{a}_i, \\ \bar{c} &= \sum_i c_i((a_i, a_i) \cdot (c, c)^{-1}) \bar{a}_i. \end{aligned}$$

This shows first that all roots in the new system are linear combinations with coefficients of the same sign of $\bar{a}_1, \dots, \bar{a}_l$; hence $\bar{a}_i > 0$ defines a Weyl chamber for the new system, and the \bar{a}_i are a simple system of roots. Therefore the coefficients of \bar{c} are integers.

4.9. *Let G be simple, U be a maximal connected subgroup of maximal rank, and b be the sum of the positive roots of U . Then (b, a) is integral for all roots a of G , the minimal root length being assumed to be 1.*

Proof. Since $W(U) \subset W(G)$, it is enough to prove this for one particular ordering. Let us consider one, say \mathcal{S} , with respect to which the simple roots of U are, in the notation of 4.3, $-d$ and the a_i 's ($i \neq j$). We have then [1, § 3.1]

$$(4) \quad (b, a_i) = (a_i, a_i) \quad (i \neq j), \quad (d, b) = -(d, d).$$

The minimal root length being assumed to be 1, these are all integers (4.1) and it is therefore sufficient to show that (b, a_j) is an integer. (4) yields

$$(5) \quad d_j(b, a_j) = -(d, d) - \sum_{i \neq j} d_i(a_i, a_i),$$

hence $d_j(b, a_j)$ is an integer. By 4.1, (b, a_j) is at any rate a half integer, so that we are done if d_j is odd. If all scalar products (a, a) are equal to 1, our assertion follows from (4) and 4.7; there remains therefore the case where $d_j = 2$ and (4.1) there are two root lengths. By 4.8, $d_i(a_i, a_i)(d, d)^{-1}$ is integral and by 4.2, d has maximal length; thus if $(d, d) = 2$, each term on the right hand side of (5) is even, and (b, a_j) is integral. If now $(d, d) = 3$, then $G = G_2$, $d = 3a_1 + 2a_2$, which implies $j = 2$, $(a_1, a_1) = 1$ and $d_2(b, a_2) = -6$.

4.10. *Let G be simple, and assume that there are two root lengths $s < t$. Then any root of length t is the sum of two roots of length s .*

Let a be a root of length t . Since $W(G)$ is irreducible, there exists at least one root b of length s , not orthogonal to a ; then $c = b - q(b, a)a$ is a root of length s (4.1). Since $q(b, a) = \pm 1$ by 4.1, our assertion is proved.

4.11. **THEOREM.** *Let U be a proper connected semi-simple subgroup of maximal rank of G and let b be the sum of the positive roots of U . Then b is singular in G .*

As in 4.6, it is first seen that it suffices to prove our assertion for some ordering, when G is simple and U is maximal connected.

Proof will be by contradiction. Assume that b is regular. Let then \mathcal{J} be the ordering of the roots of G defined by $a > 0$ if and only if $(b, a) > 0$ [1, § 2.8]. On the roots of U , it coincides with the original ordering, with respect to which b had been defined, as follows from [1, § 3.1], hence b is also the sum of the roots of U which are positive for \mathcal{J} . Let us number the simple roots a_i ($1 \leq i \leq l$) for \mathcal{J} so that a_i is a root of U if and only if $i \leq j$. (A priori, it is conceivable that no a_i belongs to U , in which case we set $j = 0$.) The $l - j$ other simple roots of U will then be denoted by a'_i ($j + 1 \leq i \leq l$). Of course $j \neq l$ since $U \neq G$. By [1, § 3.1]

$$(6) \quad (b, a_i) = (a_i, a_i) \quad (i \leq j), \quad (b, a'_i) = (a'_i, a'_i) \quad (i \geq j + 1).$$

For a given a'_i , there exist non negative integral c_j 's such that $a'_i = c_1 a_1 + \dots + c_l a_l$, therefore

$$(7) \quad (a'_i, a'_i) = (b, a'_i) = c_1 (a_1, a_1) + \dots + c_j (a_j, a_j) + c_{j+1} (b, a_{j+1}) + \dots + c_l (b, a_l).$$

At least two c_i 's are not zero; by the definition of \mathcal{J} and 4.9 we have $(a'_i, a'_i) \geq 2$, hence a'_i has maximal length.

We want to prove now that $G \neq G_2$. If G were equal to G_2 , then $a'_i = c_1 a_1 + c_2 a_2$ with $c_1 \cdot c_2 \neq 0$, $(a'_i, a'_i) = 3$, hence by 4.8 the coefficient of the root of length one would have to be a multiple of 3, but this contradicts (7) and the fact that all scalar products are integers ≥ 1 .

Thus, $G \neq G_2$, there are two root lengths, and $(a'_i, a'_i) = 2$. It also follows that two of the c_i 's, say $c_{s(i)}, c_{t(i)}$ ($s(i) < t(i)$) are equal to 1, and the others to zero. Since a'_i is simple as a root of U , we must have $t(i) \geq j + 1$, and then also $s(i) \geq j + 1$ since the root system of U is closed (4.3); (4.8) implies then that $(a_k, a_k) = 2$ for $k = s(i), t(i)$. In particular, we see that all simple roots of G of length one belong to U .

Let now c be the first (with respect to \mathcal{J}) positive complementary root of length one. This exists in view of 4.10. In order to have a contradiction, it is enough to prove that $(b, c) \leq 0$ and this will follow if we show that

$$(8) \quad (c, a_i) \leq 0 \quad (i \leq j), \quad (c, a'_i) \leq 0 \quad (i \geq j + 1).$$

By the above, c is not a simple root, therefore $c - q(c, a_i) a_i$, expressed as linear combination of the a_i ($1 \leq i \leq l$), has some positive coefficient. If $(c, a'_i) \neq 0$, then $q(c, a'_i) = \pm 1$ by 4.1 and because of $(c, c) = 1$, $(a'_i, a'_i) = 2$; since

a_i' is the sum of two simple roots, it follows again that $c - q(c, a_i')a_i'$ has some positive coefficient. Therefore, the roots $c - q(c, a_i)a_i$ ($i \leq j$), and $c - q(c, a_i')a_i'$ ($i \geq j + 1$) are positive, and moreover complementary of length one since c is. By the choice of c , they must then be greater than c , in the sense of \mathcal{B} , and this implies (8).

5. The Stiefel-Whitney class of G/T .

5.1. Let ξ be a differentiable bundle with connected fibres. Let $b \in B_\xi$ and $F = \pi_\xi^{-1}(b)$. The normal bundle to F in E_ξ is of course trivial, since it is induced by π_ξ from the tangent space of B_ξ at b . Therefore F is orientable if E_ξ is. Furthermore, the multiplication theorem [1, § 9.7] shows that $w(F)$ (resp. $\tilde{p}(F)$) is the restriction to F of $w(E_\xi)$, (resp. $\tilde{p}(E_\xi)$). In particular, it reduces to 1 if E_ξ is parallelizable. More precisely, if the S -class (§ 1) of the tangent bundle to E_ξ is zero, then the S -class of the tangent bundle to F is also zero. A similar observation is valid for Chern classes and S -equivalence class in a complex analytic (or almost complex) bundle.

5.2. Assume now that ξ is a principal differentiable bundle. The bundle along the fibres $\hat{\xi}$ [1, § 7.4] is then parallelizable. Since the tangent bundle to E_ξ is the sum of $\hat{\xi}$ and of the bundle induced by π_ξ from the tangent bundle to B_ξ [1, § 7.6], its S -class will be zero if the S -class of the tangent bundle to B_ξ is zero. Furthermore, the multiplication theorem gives

$$\pi_\xi^*(w(B_\xi)) = w(E_\xi), \quad \pi_\xi^*(\tilde{p}(B_\xi)) = \tilde{p}(E_\xi).$$

Hence if $w(B_\xi) = 1$ (resp. $\tilde{p}(B_\xi) = 1$), then $w(E_\xi) = 1$ (resp. $\tilde{p}(E_\xi) = 1$). If $w(E_\xi) = 1$ (resp. $\tilde{p}(E_\xi) = 1$), and π_ξ^* is injective, then $w(B_\xi) = 1$ (resp. $\tilde{p}(B_\xi) = 1$). (Coefficients in a field of characteristic two for the Stiefel-Whitney classes, arbitrary coefficients for the Pontrjagin classes.) Again a similar assertion is valid for Chern classes in the almost complex case.

5.3. PROPOSITION. *Let G be a compact connected Lie group, S a toral subgroup. Then $w(G/S) = 1$ and $\tilde{p}(G/S) = 1$.*

Let T be a maximal torus containing S . Then we have the principal fibering $(G/S, G/T, T/S)$, therefore (5.2) it is enough to prove 5.3 for $S = T$. For the Pontrjagin class, see [1, § 10.9]. There remains to prove that $w(G/T) = 1$. Without loss of generality it may be assumed that G is semi-simple and simply connected. Let Q be the subgroup of elements of order two in T . Then $(G/Q, G/T, T/Q, \pi)$ is a principal fibering. The total space, being the quotient of a group by a finite subgroup, is parallelizable, therefore,

(5.2) it will be enough to show that π^* is injective in cohomology mod 2. Since G and G/T are simply connected, $\pi_1(G/Q) = Q$, and the map $\pi_1(T/Q) \rightarrow \pi_1(G/Q)$ defined by the inclusion i is surjective. It follows easily that $i^*: H^*(G/Q, \mathbf{Z}_2) \rightarrow H^*(T/Q, \mathbf{Z}_2)$ is an isomorphism in dimension 1. But T/Q is a torus, hence $H^*(T/Q, \mathbf{Z}_2)$ is generated by its element of degree ≤ 1 ; therefore i^* is surjective, the fibre is totally non homologous to zero in cohomology mod 2; as is well known, this implies that π^* is injective.

5.4. It can also be derived directly from 5.1, 5.2 that *the S-class (§ 1) of the tangent bundle to G/S (S toral subgroup of G) is zero.*

In view of 5.2 and of the existence of the principal fibering $(G/S, G/T, T/S)$ it is enough to prove 5.4 when $S = T$ is a maximal torus. Let \mathfrak{g} be the Lie algebra of G , \mathcal{R} the set of regular elements, and let G operate on \mathfrak{g} by the adjoint representation. Since the centralizer of a regular element $x \in \mathfrak{g}$ is the maximal torus containing the one-parameter subgroup spanned by x , the orbits of G in \mathcal{R} are homeomorphic to G/T . Moreover, it is classical, and easily checked, that these orbits are the fibres in a differentiable fibering of \mathcal{R} . Since \mathcal{R} is parallelizable, as an open subset of \mathfrak{g} , our assertion follows from 5.1.

The nullity of $w_2(G/T)$ was noticed in [1, § 22.3] and, as remarked above, $\tilde{p}(G/T) = 1$ was also proved in [1, § 10.9].

5.5. Without entering into details, let us mention a case containing the preceding one, in which 5.1 applies. Let G operate differentiably on a connected manifold M . Among the different stability groups $G_x = \{g \in G, g \cdot x = x\}$, let H be one of smallest dimension, which has the minimal number of connected components among stability groups of that dimension. Then the set of points whose stability group is conjugate to H is an open set in M , which is differentiably fibered by the orbits [9, pp. 221-222]. Those are homeomorphic to G/H , and are called the *main orbits*. 5.1 yields then the

PROPOSITION. *Let G be a compact Lie group acting on a connected manifold M , and let F be a main orbit. Then F is orientable if M is. $w(F)$ and $\tilde{p}(F)$ are the restrictions to F of $w(M)$ and $\tilde{p}(M)$. If the S -class of the tangent bundle to M is zero, then the S -class of the tangent bundle to F is zero. In particular, if G/H is homeomorphic to the main orbit of a linear representation then it is orientable, the S -class of its tangent bundle is zero, and $w(G/H) = 1$, $\tilde{p}(G/H) = 1$.*

The proof given in 5.4 is the particular case of 5.5 corresponding to the adjoint representation, where the main orbits are homeomorphic to G/T .

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