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ON THE EULER CHARACTERISTIC OF MANIFOLDS WITH $c_1 = 0$. A LETTER TO V. GRITSENKO

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Dear Gritsenko:

The polynomial $\chi_y(X) = \sum_{p=0}^{n} \chi^p(X) y^p$ is well defined for every stably almost complex manifold ($\dim_{\mathbb{C}} X = n$). One uses the expressions for χ^p in terms of Chern numbers. It is well known that the χ^p are integers. I know from the preprint [5] by S. M. Salamon that the development of the polynomial χ_y at $y = -1$ is interesting. You told me that this also occurs in the paper [4] by A. Libgober and J. Wood. Salamon points out that the Chern number expressions for the integers

$$\frac{\chi_{-1}^{(2k)}}{(2k)!}$$

do not contain Chern classes c_i with $2k \le i \le n - 2k$. R. Jung gave me a computer output for $\chi_{-1}^{(i)}/i!$ for $i \le 10$. Denoting the number $\chi_{-1}^{(i)}/i!$ by t_i, we have

$$\begin{aligned}
t_0 &= c_n,\\
t_1 &= -nc_n/2,\\
t_2 &= (2c_1c_{n-1} + n(3n-5)c_n)/24,\\
t_3 &= -(2c_1c_{n-1} + n(n-3)c_n)(n-2)/48,\\
t_4 &= (8(-c_1^3 + 3(c_1c_2 - c_3))c_{n-3} + 8(c_1^2 + 3c_2)c_{n-2} + 4(15n^2 - 85n + 108)c_1c_{n-1}\\
&\quad + n(15n^3 - 150n^2 + 485n - 502)c_n)/5760.
\end{aligned}$$

We want to get divisibility results for the Chern number c_n under the assumption $c_1c_{n-1} = 0$ or under the stronger one: $c_1 = 0$ in integral cohomology. For this t_1, t_2, t_3 are useful.

Proposition. *For a stably almost complex manifold of dimension n,*
1) *n is odd $\Longrightarrow$ c_n is even;*
2) *if $c_1c_{n-1} = 0$, then*

$$\begin{aligned}
n(n+1)c_n &\equiv 0 \mod 8,\\
n(n-2)(n-3)c_n &\equiv 0 \mod 16;
\end{aligned}$$

3) $c_1c_{n-1} = nc_n \mod 3$.

This follows from the integrality of t_2 and t_3. For 3), see [1, Theorem 5.3*]. Therefore, for $c_1c_{n-1} = 0$ (suppose $n \ge 2$) we get the divisibility of c_n by a power of 2 depending only on the residue class of n modulo 8,

$c_1 c_{n-1} = 0 \Longrightarrow$

n mod 8	c_n is divisible by
1	8
2	4
3	2
4	2
5	8
6	4
7	4
8	1

To prove this we apply the two congruences in part 2) of the Proposition.

The above table is best possible. There exists a stably almost complex manifold W of complex dimension 8 with $c_1 = 0$ and c_8 odd. (See [6, p. 278]: the Cayley projective plane (with the Stiefel–Whitney number $w_{16} = 1$) as unoriented bordism class is in the image of Ω_{16}^{SU}.) For $n \equiv 1 \mod 8$ take $(S^6)^3 \times W^i$. For $n \equiv 2 \mod 8$ take the Enriques surface ($c_2 = 12$) and multiply it with W^i. For $n \equiv 3 \mod 8$ use $S^6 \times W^i$. For $n = 4$ observe that the smooth hypersurface Y of degree 6 in $P_5(\mathbb{C})$ has Euler number $\equiv 2 \mod 4$. For $n = 5$ multiply the Enriques surface with S^6. For $n = 6, 7$ use $(S^6)^2$ and $Y \times S^6$, respectively.

By part 3) of the Proposition, for n mod 3 the vanishing of $c_1 c_{n-1}$ implies that

$$c_n \equiv 0 \mod 3 \quad \text{if } n \not\equiv 0 \mod 3.$$

If K is a $K3$-surface, then $K \times (S^6)^i$ and $(S^6)^i$ show that this result is best possible for $n \equiv 0, 2 \mod 3$, even for $c_1 = 0$. The quadric in $P_5(\mathbb{C})$ carries a stably almost complex structure with $c_1 = 0$ and $c_4 = 6$ ([3]; has to be checked). The result is therefore best possible for $n \equiv 1 \mod 3$, even for $c_1 = 0$.

Proposition. *If X is a stably almost complex manifold with $c_1 = 0$ and even complex dimension $n = 2k$ with $k \equiv 1 \mod 4$, then $c_n \equiv 0 \mod 8$.*

Remark. In the above table, if $c_1 = 0$, for $n = 2 \mod 8$, the divisibility by 4 must be replaced with that by 8. The result in the table is then sharp. We see this by replacing the Enriques surface by a $K3$-surface.

For the proof we note that

$$\chi_y = t_0 + t_1(1+y) + t_2(1+y)^2 + \cdots .$$

For $y = 1$ we obtain the signature,

$$\text{sign} \equiv t_0 + 2t_1 + 4t_2 + 8t_3 \mod 16.$$

If $c_1 c_{n-1} = 0$, this implies the relation

$$\text{sign} \equiv c_n \left[1 - n + \frac{n(3n-5)}{6} - \frac{n(n-2)(n-3)}{6} \right] \mod 16.$$

For $n = 2k$ we have

$$\text{sign} \equiv c_{2k} \left[\frac{3 - k - 4k^3}{3} \right] \mod 16.$$

For k even

$$\text{sign} = c_{2k}(5k+1) \mod 16,$$

and for k odd

$$(*) \qquad \text{sign} \equiv c_{2k}(9k+1) \mod 16.$$

For k odd, c_{2k} is divisible by 4. Hence,

$$\text{sign} \equiv 0 \mod 8,$$

and

$$\text{sign} \equiv 0 \mod 16 \quad \text{for } k \equiv 3 \mod 4.$$

Now we recall the following theorem of Serge Ochanine (cf. [2, p. 113]).

Theorem. *For a 4k-dimensional Spin-manifold (k odd) the signature is divisible by* 16.

A stably almost complex manifold with $c_1 = 0$ is Spin. We have proved the divisibility of the signature by 8 (respectively, 16) for $k \equiv 1 \mod 4$ (respectively, $k \equiv 3 \mod 4$) under the weaker assumption $c_1 c_{2k-1} \equiv 0$. However, we wanted to prove something else. Namely, $c_{2k} \equiv 0 \mod 8$ for $k \equiv 1 \mod 4$ under the assumption $c_1 = 0$. We look again at $(*)$ for $k \equiv 1 \mod 4$. The number $9k+1$ is $\equiv 2 \mod 4$. Hence, by Ochanine, $c_{2k} \equiv 0 \mod 8$.

I lectured about these results in Warsaw (June 1994) and in Potsdam (October 1995).

With best regards,

F. Hirzebruch

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