THE RING OF HILBERT MODULAR FORMS FOR REAL QUADRATIC FIELDS OF SMALL DISCRIMINANT

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In this lecture we shall show how the resolution of the singularities at the cusps of the Hilbert modular surfaces [7] can be used for a detailed study of these surfaces which makes it possible in some cases to determine the structure of the ring of Hilbert modular forms.
§1. CUSP SINGULARITIES AND INVOLUTIONS.

Let $K$ be a real quadratic field, M C K a module (free $\mathbb{Z}$-module of rank 2) and $U_{M}^{+}$the group of the totally positive units $\varepsilon$ of $K$ with $\varepsilon M=M$. The group $U_{M}^{+}$is infinite cyclic. Let $V \subset U_{M}^{+}$be a subgroup of finite index. The semi-direct product

$$
G(M, V)=\left\{\left.\left(\begin{array}{ll}
\varepsilon & \mu \\
0 & 1
\end{array}\right) \right\rvert\, \varepsilon \in V, \mu \in M\right\}
$$

acts freely on $H^{2}$ by

$$
\left(z_{1}, z_{2}\right) \mapsto\left(\varepsilon z_{1}+\mu, \varepsilon^{\prime} z_{2}+\mu^{\prime}\right),
$$

where $x \mapsto x^{\prime}$ is the non-trivial automorphism of $K$. We add a point to $H^{2} / G(M, V)$ and topologize $H^{2} / G(M, V) \cup\{\infty\}$ by taking

$$
\left\{\left(z_{1}, z_{2}\right) \in H^{2} \mid y_{1} y_{2}>c\right\} / G(M, v) \cup\{\infty\}
$$

(for $C>0$ ) as neighborhoods of $\infty$. (Notation : $z_{j}=x_{j}+i y_{j}$ with
$x_{j}, y_{j} \in \mathbb{R}$ and $\left.y_{j}>0\right)$. Then $H^{2} / G(M, V) U\{\infty\}$ is a normal complex space with $\infty$ as the only singular point. This is the "cusp singularity" defined by $M, V$. The local ring at $\infty$ is denoted by $O(M, V)$. It is the ring of all Fourier series $f$ convergent in some neighborhood of $\infty$ of the form

$$
\begin{gather*}
f=a_{0}+\sum_{\substack{\lambda \in M^{*} \\
\lambda \gg 0 \\
a_{\lambda}=a_{\varepsilon \lambda}}} a_{\lambda} \cdot e^{2 \pi i\left(\lambda z_{1}+\lambda^{\prime} z_{2}\right)} \varepsilon \in V \tag{1}
\end{gather*}
$$

where $M^{*}$ is the dual module of $M$, i.e.

$$
M^{*}=\{\lambda \in \mathbb{K} \mid \operatorname{Tr}(\lambda \mu) \in \mathbb{Z} \quad \text { for all } \mu \in M\}
$$

The singular point $\infty$ can be resolved [7]. Under the process of minimal desingularisation it is blown up into a cycle of $r$ non-singular rational curves ( $r \geqq 2$ ) or into one rational curve with a double point ( $r=1$ ). Such a cycle is indicated by a diagram

where $-b_{0},-b_{1}, \ldots$ are the selfintersection-numbers (for $r \geq 2$ ). We have $b_{i} \geqq 2$. This cycle of numbers is denoted by ( $\left(b_{0}, b_{1}, \ldots, b_{r-1}\right)$ ). It is determined by the denominators of a periodic continued fraction associated to $M$, see [7].

The non-singular surface obtained from $H^{2} / G(M, V) \cup\{\infty\}$ by resolving the singular point will be called $X(M, V)$. Of course, it is not compact. For the intersection point of two consecutive curves of the cycle we
have a natural coordinate system (u,v) centered at that point [7]. Any $f \in O(M, V)$ can be written as a power series in $u, v$ (this is analogous to the q-expansion in one variable.)

If $M=M^{\prime}$, then the cusp is called symmetric. The involution $\tau:\left(z_{1}, z_{2}\right) \mapsto\left(z_{2}, z_{1}\right)$ operates on $H^{2} / G(M, V) \cup\{\infty\}$ with $\tau(\infty)=\infty$. The fixed point set of $\tau$ in $H^{2} / G(M, V)$ is a non-singular curve $C$ consisting of 2,3 or 4 components. Therefore, the quotient of $H^{2} / G(M, V) \cup\{\infty\}$ by $\tau$ has no singular point except possibly $\infty_{\tau}$, the image of $\infty$. The involution $\tau$ acts on $O(M, V)$, and $O(M, V)^{\tau}$ (consisting of all $f$ in (1) with $a_{\lambda}=a_{\lambda}$, for $\lambda \in M$ ) is the local ring at $\infty_{\tau}$. The involution $\tau$ can be lifted to $X(M, V)$. The curve $C$ extends to a non-singular curve in $X(M, V)$, also denoted by $C$. If the number of curves in the cycle is even, then T maps none or two of the curves in the cycle, say S and T , to themselves and interchanges the others. If the number of curves in the cycle is odd, then $\tau$ maps one curve $S$ in the cycle to itself and interchanges the others. The curve $C$ intersects each curve $S$ and $T$ transversally in two points or in one point depending on whether the selfintersection number of $S$ or $T$ respectively is even or odd. The fixed point set of $\tau$ in $X(M, V)$ consists of $C$ and an isolated fixed point on each of the curves $S$ and $T$ which have odd selfintersectionnumber. Blowing up the isolated fixed points of t gives a surface $\widetilde{X}(M, V)$ on which $\tau$ operates having no isolated fixed points. The exceptional curves on $\tilde{X}(M, V)$ obtained by this blowing up belong to the fixed point set of $\tau$. The surface $\widetilde{X}(M, V) / \tau$ is non-singular. On it we have a chain of rational curves mapping to $\infty_{\tau}$. This is a resolution of $\infty_{\tau}$. It need not be minimal. In fact, $\infty_{\tau}$ could be a regular point. In any case, the existence of this resolution by a chain of rational curves proves that ${ }_{\tau}^{\infty}$ is a quotient singularity [6], [1]. The above investigation of $X(M, V)$ for $M=M^{\prime}$ is due to Karras [12] (Lemma 3.3). The fact that ${ }_{\tau}$ is a quotient singularity was proved earlier by H . Cohn
and E. Freitag (see the Iiterature quoted in [12]). Gundlach [5] has given necessary and sufficient conditions that ${ }^{\infty}{ }_{\tau}$ is regular. Such symmetric cusps are called quasi regular.

THEOREM (Karras). A cusp given by ( $M, V$ with $M=M^{\prime}$ is quasi regular if and only if its cycle $\left(\left(b_{0}, b_{1}, \ldots, b_{r-1}\right)\right)$ is equal to one of the following cycles

$$
\begin{array}{ll}
\text { i) }((3, \underbrace{2, \ldots, 2}_{m})) & \text { with } m \geqq 0 \\
\text { ii) }((4, \underbrace{2, \ldots, 2)}_{m}) & \text { with } m \geqq 0 \\
\text { iii) }(\underbrace{2}_{\underbrace{2}_{m}, \ldots, 2}, 3, \underbrace{2, \ldots, \underbrace{2}_{m}}_{\mathrm{n}}, 3)) & \text { with } m \geqq n
\end{array}
$$

and if in iii) the two curves of selfintersection number -3 are interchanged under $\tau$ (which is automatic for $m>n$ ).

Consider the following curves in $C^{2}$ (coordinates $X, Y$ )
i) $\left(X+Y^{2}\right)\left(X^{2}+Y^{m+5}\right)=0 \quad$ with $m \geq 0$
ii) $\left(X^{2}+Y^{2}\right)\left(X^{2}+Y^{m+3}\right)=0 \quad$ with $m \geqq 0$
iii) $\left(X^{n+3}+Y^{2}\right)\left(X^{2}+Y^{m+3}\right)=0 \quad$ with $m \geqq n \geqq 0$

Let $F(X, Y)=0$ be one of these curves. The double cover of $\mathbb{C}^{2}$ branced along $F(X, Y)=0$ has the point above $(0,0) \in \mathbb{C}^{2}$ as isolated singular point whose minimal resolution is a cycle of rational curves with selfintersection numbers as given in the preceding theorem of Karras. This can be checked directly. By a theorem of Laufer [15] (see also [13]) a singularity whose resolution is a cycle of rational curves is determined up to biholomorphic equivalence by its cycle of selfintersection numbers. Therefore, the structure of the local rings $O(M, V)$ of quasi regular cusps is now known ([12], Satz 3), namely

$$
O(M, V) \cong \mathbb{C}[C, Y, Z] /\left(Z^{2}=F(X, Y)\right)
$$

where $F(X, Y)$ is the polynomial given in i), ii), iii) above and where $\tau$ corresponds to the natural involution of the double cover. See also H. Cohn as quoted in [12].

In the following examples a), b), c) of quasi regular cusps we indicate the fixed point set $C$ of $\tau$ on $X(M, V)$ by heavily drawn lines. Isolated fixed points of $\tau$ on $X(M, V)$ do not occur in examples $a), b), c)$.
a)

b)

c)


In example a) we have $K=\mathbb{Q}(\sqrt{5})$ with $M=\sqrt{5} .0$ and $\left[U_{M}^{+}\right.$: V] $=2$. (For a field $K$ we denote its ring of integers by 0. .) After dividing by $\tau$ (which interchanges the two (-3)-curves) we have in $X(M, V) / \tau$ the following situation
a)


The non-singular rational (-1)-curve is the image of the two (-3)-curves. The image curve of $C$ will also be denoted by $C$. It simply touches the ( -1 )-curve in two points. If we blow down the ( -1 )-curve we get $\left(H^{2} / G(M, V)\right) / \tau \cup\left\{\infty_{\tau}\right\}$ which shows that $\infty_{\tau}$ is regular. After blowing down the (-1)-curve, the two components of $C$ become singular. Each has a cusp (in the sense of curve singularities). The two cusps have separate tangents which checks with iii) ( $m=n=0$ ). The structure of $O(M, V)$ is given by (2). Therefore, there must exist three Fourier series $f, g, h$ as in (1) generating $O(M, V)$ and satisfying $h^{2}=\left(f^{3}+g^{2}\right)\left(f^{2}+g^{3}\right)$.

In example b) we have $K=Q(\sqrt{2})$ with $M=0$ and $V=U_{M}^{+}$

in $X(M, V)$

in $X(M, V) / \tau$

We have numbered the four branches of $C$.
In $X(M, V) / \tau$ we blow down the $(-1)$-curve, the $(-2)$-curve becomes a (-1)curve and can be blown down also. The image of the two curves is $\infty_{\tau}$, which is therefore a regular point. In $\left(H^{2} / G(M, V)\right) / \tau U\{\infty\}$ the foun branches of $C$ in a neighborhood of $\infty_{\tau}$ behave as follows:
b)

$C_{3}, C_{4}$ touch simply, all other intersections are transversal. This checks with ii) ( $m=1$ ).

In example c) we have $K=Q(\sqrt{7})$ with $M=\sqrt{7} .0$ and $V=U_{M}^{+}$.
c)


in $X(M, V)$
in $X(M, V) / \tau$

In $X(M, V) / \tau$ the ( -2 )-curve touches the component $C_{3}$ of $C$ simply. Blowing down ${ }^{-1} \times-2 \times-2$ gives the regular point $\infty_{T}$ where $c_{1}, c_{2}, c_{3}$ behave locally like

$$
\left(X^{3}+Y^{2}\right)\left(X^{2}+Y^{6}\right)=0
$$

with $X^{3}+Y^{2}=0$ corresponding to $C_{3}$, and $X \pm i . Y^{3}=0$ to $C_{1}$ and $C_{2}$ respectively (compare iii), $n=0, m=3$ ).

The following symmetric cusp is not quasi regular.
d)

- = isolated fixed point of $\tau$


We have $K=\mathbb{Q}(\sqrt{13})$ with $M=0$ and $\left[U_{M}^{+}: V\right]=3$. Before dividing by $\tau$ we blow up the isolated fixed point. Then we divide by $\tau$ and obtain a configuration

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which after blowing down the (-1)-curve shows that ${ }^{\infty}{ }_{\tau}$ is a quotien singularity admitting the minimal resolution


Thus it is the quotient singularity of type (36; 11,1), see [6].

## §2. THE DIMENSION FORMULA FOR HILBERT CUSP FORMS.

Let $K$ be a real quadratic field and $O$ the ring of integers of $K$. The Hilbert modular group $\mathrm{SL}_{2}(0) /\{ \pm 1\}$ operates effectively on $\mathrm{H}^{2}$. According to siegel the volume of $\mathrm{H}^{2} / \mathrm{SL}_{2}(0)$ equals $2 \zeta_{\mathrm{K}}(-1)$. The volume is normalized such that if T is a subgroup of $\mathrm{SL}_{2}(0) /\{ \pm 1\}$ of finite index a which acts freely on $H^{2}$, then

$$
\begin{equation*}
\operatorname{vol}\left(H^{2} / \Gamma\right)=2 \zeta_{K}(-1) \cdot a=e\left(H^{2} / G\right) \tag{3}
\end{equation*}
$$

where $e(A)$ denotes the Euler number of the space $A$. (Though $H^{2} / \Gamma$ is non-compact, the Euler number can be calculated by the volume, this is a special case of a result of Harder, see [7] and the literature quoted there.)

Let $S_{k}(\Gamma)$ be the complex vector space of cusp forms of weight $k$ for $\Gamma$ where $\Gamma$ is a subgroup of $\operatorname{SL}_{2}(0) /\{ \pm 1\}$ of finite index.

The weight $k$ of $a$ form $f$ is defined by the transformation law

$$
f\left(\frac{a z_{1}+b}{c z_{1}+d}, \frac{a^{\prime} z_{2}+b^{\prime}}{c^{\prime} z_{2}+c^{\prime}}\right)=\left(c z_{1}+d\right)^{k}\left(c^{\prime} z_{2}+d^{\prime}\right)^{k} f\left(z_{1}, z_{2}\right)
$$

This is well-defined also for $k$ odd, because the expression $\left(c z_{1}+d\right)^{k}\left(c^{\prime} z_{2}+d^{\prime}\right)^{k}$ does not change if $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ is replaced by $\left(\begin{array}{ll}-a & -b \\ -c & -d\end{array}\right)$.

THEOREM. If $\Gamma$ has index $a$ in $S_{2}(0) /\{ \pm 1\}$ and acts freely on $H^{2}$, then for $k \geqq 3$
(4)

$$
\begin{aligned}
\operatorname{dim} S_{k}(\Gamma) & =\frac{k(k-2)}{2} \zeta_{K}(-1) \cdot a+\chi \\
& =\frac{k(k-2)}{4} e\left(H^{2} / \Gamma\right)+\chi
\end{aligned}
$$

where $x=1+\operatorname{dim} S_{2}(\Gamma)$.

The formula (4) is found in the literature only for $k$ even. But it seems to be known also for odd $k$ (see Shimizu [17], p. 63, footnote 11). On my request, $H$. Saito has checked that (4) can be proved for odd $k$ in the same way as Shimizu does it.

If $\Gamma$ acts freely, then $H^{2} / \Gamma$ is a non-singular complex surface which can be compactified by finitely many points, the cusps, to give a compact surface $\overline{H^{2} / \Gamma}$. The isotropy groups of the cusps are of the form $G(M, V)$. The cusps are singular points of $\overline{\mathrm{H}^{2} / \mathrm{T}}$ and can be resolved in the minimal canonical way as recalled in $\$ 1$. The resulting surface is a nonsingular algebraic surface $Y(\Gamma)$. It is a regular surface, i.e. its first Betti number vanishes, but it is not necessarily simply-connected. The cusp forms of weight 2 can be extended to holomonphic differential forms on $Y(\Gamma)$ (sections of the canonical bundle of $Y(\Gamma)$ ). Therefore, $\operatorname{dim} S_{2}(\Gamma)$ is the geometric genus $\mathrm{p}_{\mathrm{g}}$ of $Y(\Gamma)$ and $X$ the arithmetic genus. The fact that the constant tem in the Shimizu polynomial (4) is the arithmetic genus of $Y(\Gamma)$ was discovered by Freitag (compare [7], 3.6.).
§3. THE FIELD $K=\mathbb{Q}(\sqrt{5})$.

In the field $K=\mathbb{Q}(\sqrt{5})$ the ring 0 of integers consists of all linear combinations $a+b(1+\sqrt{5}) / 2$ with $a, b \in \mathbb{Z}$. To the prime ideal generated in 0 by $\sqrt{5}$ there belongs a principal congruence subgroup of $\mathrm{SL}_{2}(0)$, which we denote by r .

$$
\Gamma=\left\{\left.\left(\begin{array}{ll}
\alpha & \beta \\
\gamma & \delta
\end{array}\right) \in \operatorname{SL}_{2}(0) \right\rvert\, \alpha \equiv \delta \equiv 1(\bmod \sqrt{5}), \beta \equiv \gamma \equiv 0(\bmod \sqrt{5})\right\} .
$$

Because $\left(\begin{array}{cc}-1 & 0 \\ 0 & -1\end{array}\right) \notin \Gamma$, the group $\Gamma$ can be regarded as a subgroup of the Hilbert modular group $G=S L_{2}(0) /\{ \pm 1\}=\operatorname{PSL}_{2}(0)$. The group $\Gamma$ acts freely on $\mathrm{H}^{2}$. The volume of $\mathrm{H}^{2} / \mathrm{G}$ is equal to $2 \zeta_{\mathrm{K}}(-1)=1 / 15$. The factor group $G / \Gamma$ is isomorphic to $\mathrm{PSL}_{2}\left(\mathbb{F}_{5}\right)$ because $0 / \sqrt{5} .0 \cong \mathbb{F}_{5}$. In
its turn, $\operatorname{PSL}_{2}\left(\mathbb{F}_{5}\right)$ is isomorphic to the alternating group $A_{5}$. Namely, $A_{5}$ is the group of automorphisms of the icosahedron and acts on the six axes of the icosahedron through its vertices in the same way as $\mathrm{PSL}_{2}\left(\mathbb{F}_{5}\right)$ acts on the six points of the projective line $\mathrm{P}_{1}\left(\mathbb{F}_{5}\right)$. We have

$$
\begin{equation*}
e\left(H^{2} / \Gamma\right)=\left|A_{5}\right| \cdot 2 \zeta_{K}(-1)=60 \cdot \frac{1}{15}=4 \tag{5}
\end{equation*}
$$

The space $\mathrm{H}^{2} / \Gamma$ is compactified by adding six cusps. Since the class number of $Q(\sqrt{5})$ is 1 , the action of $G$ on $P_{1}(K)=K U\{\infty\}$ has only one orbit, while the action of $\Gamma$ on $P_{1}(k)$ has six. This follows, because the isotropy group of $G$ and $\Gamma$ at $\infty$ satisfy $\left|G_{\infty} / \Gamma_{\infty}\right|=10$. In fact, $G_{\infty} / \Gamma_{\infty}$ is the dihedral group of order 10 , this will be used later. Two points $\alpha / \delta$ and $\gamma / \delta$ in $P_{1}(K)$ with $\alpha, \beta, \gamma, \delta \in O$ and $(\alpha, \beta)=(\gamma, \delta)=1$ belong to the same orbit precisely when $\alpha \equiv \gamma(\bmod \sqrt{5})$ and $\beta \equiv \delta$ $(\bmod \sqrt{5})$, that is when $\alpha / \beta$ and $\gamma / \delta$ represent the same point of $P_{1}\left(\mathbb{F}_{5}\right)$. The surface $H^{2} / \Gamma$, compactified by six points, is denoted by $\overline{H^{2} / \Gamma}$. This is an algebraic surface with $V_{\text {singular points corresponding to the six }}$ cusps. Since the action of $G$ on $H^{2}$ induces an action of $A_{S} \cong G / \Gamma$ on $\overline{H^{2} / \Gamma}$ which acts transitively on the cusps, these six singular points have the same structure, and it is sufficient to investigate the structure of the singularity at $\infty=1 / 0$. The isotropy group of $\Gamma$ at this point has the form
(6)

$$
\Gamma_{\infty}=\left\{\left.\left(\begin{array}{cc}
\varepsilon & \mu \\
0 & \varepsilon^{-1}
\end{array}\right) \right\rvert\, \varepsilon \text { unit in } 0, \varepsilon \equiv 1(\bmod \sqrt{5}), \mu \equiv 0(\bmod \sqrt{5})\right\}
$$

The fundamental unit of 0 is $\varepsilon_{0}=(1+\sqrt{5}) / 2$. The condition $\varepsilon \equiv 1$ (mod $\sqrt{5}$ ) means that $\varepsilon$ must be a power of $-\varepsilon_{0}^{2}$. The group $r_{\infty}$ can also be written as $G(M, V)$ where $M=\sqrt{5} .0$ and $V$ is generated by $\varepsilon_{0}^{4}$. Thus $\left[U_{M}^{+}: V\right]=2$ and $G(M, V)$ is as in example a) of $\$ 1$. On the surface $Y=Y(\Gamma)$ that arises from $\overline{H^{2} / \Gamma}$ by resolution of the six singular points there are six pairwise disjoint configurations
(7)


As a 4-dimensional manifold, $Y$ can be obtained as follows : $\mathrm{H}^{2} / \Gamma$ has asdeformation retract a compact manifold X whose boundary has six components. Each boundary component is a torus bundle over a circle. All boundary components are isomorphic. Every configuration (7) in Y has a tubular neighborhood having as boundary such a torus bundle. The manifold $Y$ arises by glueing to $X$ the tubular neighborhoods of these six configurations (7). Since the Euler number of each tubular neighborhood is 2, it follows from (5) and the additivity of $e$ that

$$
\begin{equation*}
e(Y)=e(X)+6.2=e\left(H^{2} / \Gamma\right)+12=16 \tag{8}
\end{equation*}
$$

The action of $A_{5}$ on $H^{2} / \Gamma$ described above induces an action on $Y$. The diagonal $z_{1}=z_{2}$ of $H^{2}$ yields a curve in $H^{2}$ i $\Gamma$, which can be compactified to a curve $C$ in $Y$. The subgroup of $I$ carrying the diagonal into itself is the ordinary principal congruence subgroup $\Gamma(5)$ of $\mathrm{SL}_{2}(\mathbb{Z})$, which can also be regarded as subgroup of $S L_{2}(\mathbb{Z}) /\{ \pm 1\}$, the quotient group being $A_{5}$ again. Therefore, each element of $A_{5}$ when acting on $Y$ carries $C$ to itself. The curve $H / P(5)$ has normalized Euler volume $-\frac{1}{6} \cdot 60=-10$ and twelve cusps. The compactified curve $\overline{H / T}(5)$ has Euler number $-10+12=2$, thus is a rational curve which maps onto $C$. For reasons of symmetry, the curve $C$ must pass through each of the six configurations (7) exactly twice. We now describe how the curve cuts a resolution (7) by reducing the question to the corresponding question for the diagonal in $H^{2} / G=\left(H^{2} / \Gamma\right) / A_{5}$. There is an exact sequence
(9)

$$
0 \rightarrow 0 / \sqrt{5} .0 \rightarrow G_{\infty} / \Gamma_{\infty} \rightarrow \mathrm{U}_{\mathrm{M}}^{+} / \mathrm{V} \rightarrow 1
$$

The groups $0 / \sqrt{5} .0$ and $U_{M}^{+} / V$ are cyclic of onder 5 and 2 respectively, and $G_{\infty} / \Gamma_{\infty}$ is a semi-direct product, namely the dihedral group of onder 10.

To understand the formation of the quotient of the configuration (7) by this dihedral group, we check first that any non-trivial element g of $0 / \sqrt{ } 5.0$ carries each of the two ( -3 )-curves to itself and has their intersection points as isolated fixed points. By blowing up these two points we come to the following configuration :

(the verticals are fixed lines for $g$ )

After factorizing by $0 / \sqrt{5} .0$ we obtain


The group $U_{M}^{+} / V \cong \mathbb{Z} / 2 \mathbb{Z}$ acts on this quotient by "rotation", carrying each (-1)-curve to the other one, each (-5)-curve to the other one.

Factorization leads to

and blowing down the (-1)-curve gives a configuration consisting of a rational curve with a double point. This is the resolution of the cusp of $\overline{H^{2} / G}$. The curve in the desingularized compactification of $H^{2} / G$ represented by $z_{1}=z_{2}$ is usually called $F_{1}$ (see [10]). It passes transversally through the resolved cusp as follows

(see [7], 84.)

As explained the configuration (7) is a ten-fold covering of (10). We conclude that $C$ passes through each configuration (7) in the two "corners" and meets in these two points each (-3)-curve of the configuration (7) transversally. This is illustrated in the following diagram
(11)


The curve $C$ is non-singular, because of the described behaviour at the cusps of $\overline{\mathrm{H}^{2} / \Gamma}$ and because two curves on $\mathrm{H}^{2}$ equivalent to the diagonal $z_{1}=z_{2}$ under $S L_{2}(0)$ cannot intersect in $H^{2}$ (see [11], 3.4. or [10]). Therefore $\overline{H / T(5)} \rightarrow \mathrm{C}$ is bijective. The value of the first Chern class $c_{1}$ of $Y$ on $C$ equals twice the Euler volume of $H / \Gamma(5)$ (which is -10 ) plus 24 (see [7], 4.3. (19)). Thus we have in $Y$

$$
\begin{equation*}
c_{1}[c]=4 \text { and } c \cdot c=2 \text { (by the adjunction fommla). } \tag{12}
\end{equation*}
$$

Because $Y$ is regular, this implies that $Y$ is a rational surface (compare [9], [7]).

The curve $\lambda z_{2}-\lambda^{\prime} z_{1}=0$ in $H^{2}$ with $\lambda=\sqrt{5} \cdot \varepsilon_{0}$ is a skew-hermitian curve which determines the curve $F_{5}$ in $H^{2} / G$ (see [10]). The inverse image $D$ of $F_{5}$ in $H^{2} / \Gamma$ consists of 15 connectedness components. Namely, as can
be checked, the subgroup of $A_{5}=G / \Gamma$ which carries the curve in $H^{2} / \Gamma$ given by $\lambda z_{2}-\lambda \prime z_{1}=0$ to itself is of order 4 . The curve $F_{5}$ passes through the resolved cusp of $H^{2} / 6$ as follows


Therefore $D$ intersects each configuration (7) in the following way


A component of $D$ intersects exactiy two of the configurations (7) and each in two points, one intersection point on each (-3)-curve. It is easy to see that each component of $D$ is a non-singular rational curve.

The involution $\left(z_{1}, z_{2}\right) \rightarrow\left(z_{2}, z_{1}\right)$ on $H^{2}$ induces an involution $\tau$ on $H^{2} / \Gamma$, because $A^{\prime} \in \Gamma$ if $A \in \Gamma$. The involution $\tau$ keeps every cusp of $H^{2} / \Gamma$ fixed, because it operates on $P_{1}(K)$ by conjugation ( $x \mapsto x^{\prime}$ ) and the cusps can be represented by rational points. Each cusp is symmetric, $\Gamma$ operates on each of the configurations (7) by interchanging the two (-3)-curves. The curve $C$ is pointwise fixed under $\tau$. In fact, $C$ is the complete fixed point set. This can be seen as follows. The involution $\left(z_{1}, z_{2}\right) \mapsto\left(z_{2}, z_{1}\right)$ induces an involution on $H^{2} / G$ which has $F_{1} \cup F_{5}$ as fixed point set ([7], §4.).
Therefore, the fixed point set of $\tau$ on $H^{2} / \Gamma$ is at most $C U D$. The inter section behaviour of such a component $D_{j}$ with a configuration (7) shows that $D_{j}$ is carried to itself under $\tau$, but is not pointwise fixed.

The quotient $Y / \tau$ is a non-singular algebraic surface. We have

$$
\begin{equation*}
e(Y / \tau)=\frac{1}{2}(e(Y)+e(C))=\frac{1}{2}(16+2)=9 \tag{13}
\end{equation*}
$$

By example a) in $\$ 1$, the surface $Y / \tau$ has six exceptional curves. If we blow them down, we get an algebraic surface $Y_{0}$ with $e\left(Y_{0}\right)=3$ and six distinguished points $P_{1}, \ldots, P_{6}$ resulting from the exceptional curves. Since $Y$ is rational, $Y / \tau$ and $Y_{0}$ are rational. Thus $Y_{0}$ is the complex projective plane, and the image of $C$ on $Y_{0}$ is a rational curve with a double cusp in each point $P_{j}(j=1, \ldots, 6)$ and is otherwise nonsingular. "Double cusp in $\mathrm{P}_{\mathrm{j}}$ " means that the curve has two branches in $P_{j}$, each with a cusp, the two cusps having separate tangents. We denote the image of $C$ in $Y_{0}$ also by $C$. Each double cusp reduces the genus in the Plücker formula by 6 . Thus the degree $n$ of $C$ in $Y_{0}=P_{2}(\mathbb{L})$ satisfies

$$
\frac{(n-1)(n-2)}{2}-6.6=0
$$

Therefore $C$ is a curve of degree 10 in $P_{2}(\mathbb{C})$, as can also be infered from (12). The image of $D$ in $Y_{0}=P_{2}(\mathbb{C})$ (also denoted by $D$ ) is the union of the 15 lines joining $P_{1}, \ldots, P_{6}$, as can be proved in a similar way.

The involution $\tau$ operating on $Y$ commutes with each element of $G / \Gamma \cong A_{5}$. This follows from the fact that matrices $A, A^{\prime} \in S L_{2}(0)$ are equivalent $\bmod \sqrt{5}$. Therefore, $A_{5}$ acts effectively on $Y / \tau$ and on $Y_{0}=P_{2}(\mathbb{C})$.

Every action of $A_{5}$ on $P_{2}(\mathbb{C})$ can be lifted to a 3-dimensional linear representation, because $\mathrm{H}^{2}\left(\mathrm{~A}_{5}, \mathbb{Z}_{3}\right)=0$.
[I. Naruki has shown me a proof that $H^{2}\left(G, \mathbb{Z}_{3}\right)=0$ for a non abelian, finite simple group $G$ whose order is not divisible by 9 . Such results essentially can be found in Schur's papers.]

The lifting is unique, because $A_{5}$ is simple. The character table shows that there are exactly two equivalence classes of non-trivial 3-dimensional representations of $A_{5}$. They are related by an outer automorphism of $A_{5}$. Hence the action of $A_{5}$ on $P_{2}(\mathbb{C})$ which we have found is essentially the one whose invariant theory was studied by F. Klein [14]. We recall some of Klein's results.

The group $A_{5}$ is isomorphic to the finite group $I$ of those elements of So(3) which carry a given icosahedron centered at the origin of the standard Euclidean space $\mathbb{R}^{3}$ to itself. The group I operates linearly on $\mathbb{R}^{3}$ (standard coordinates $x_{0}, x_{1}, x_{2}$ ) and thus also on $P_{2}(\mathbb{R})$ and $P_{2}(\mathbb{C})$. We are concerned with the action on $P_{2}(\mathbb{C})$. A curve in $P_{2}(\mathbb{C})$ which is mapped to itself by all elements of $I$ is given by a homogeneous polynomial in $x_{0}, x_{1}, x_{2}$ which is I-invariant up to constant factors and hence $I$-invariant, because $I$ is a simple group. The graded ring of all I-invariant polynomials in $x_{0}, x_{1}, x_{2}$ is generated by homogeneous polynomials $A, B, C, D$ of degrees $2,6,10,15$ with $A=x_{0}^{2}+x_{1}^{2}+x_{2}^{2}$. The action of $I$ On $P_{2}(\mathbb{C})$ has exactly one minimal orbit where "minimal" means that the number of points in the orbit is minimal. Whis orbit has six points, they are called poles. These are the points of $P_{2}(\mathbb{R}) \subset \mathrm{P}_{2}(\mathbb{D})$ which are represented by the six axes through the vertices of the icosahedron. Klein uses coordinates

$$
A_{0}=x_{0}, \quad A_{1}=x_{1}+i x_{2}, \quad A_{2}=x_{1}-i x_{2}
$$

and puts the icosahedron in such a position that the six poles are given by

$$
\begin{aligned}
& \left(A_{0}, A_{1}, A_{2}\right)=(\sqrt{5} / 2,0,0) \\
& \left(A_{0}, A_{1}, A_{2}\right)=\left(\frac{1}{2}, \varepsilon^{v}, \varepsilon^{-v}\right)
\end{aligned}
$$

with $\varepsilon=\exp (2 \pi i / 5)$ and $0 \leqq v \leqq 4$.

The invariant curve $A=0$ does not pass through the poles. There is exactly one invariant curve $B=0$ of degree 6 which passes through the poles, exactly one invariant curve $C=0$ of degree 10 which has higher multiplicity than the curve $B=0$ in the poles and exactly one invariant curve $D=0$ of degree 15. In fact, $B=0$ has an ondinary double point (multiplicity 2) in each pole, $C=0$ has a double cusp (multiplicity 4) in each pole and $D=0$ is the union of the 15 lines connecting the six poles. Klein gives formulas fon the homogeneous polynomials $A, B, C, D$ (determined up to constant factors). They generate the ring of all Iinvariant polynomials. We list Klein's formulas :

$$
\begin{aligned}
A= & A_{0}^{2}+A_{1} A_{2} \\
B= & 8 A_{0}^{4} A_{1} A_{2}-2 A_{0}^{2} A_{1}^{2} A_{2}^{2}+A_{1}^{3} A_{2}^{3}-A_{0}\left(A_{1}^{5}+A_{2}^{5}\right) \\
C= & 320 A_{0}^{6} A_{1}^{2} A_{2}^{2}-160 A_{0}^{4} A_{1}^{3} A_{2}^{3}+20 A_{0}^{2} A_{1}^{4} A_{2}^{4}+6 A_{1}^{5} A_{2}^{5} \\
& -4 A_{0}\left(A_{1}^{5}+A_{2}^{5}\right)\left(32 A_{0}^{4}-20 A_{0}^{2} A_{1} A_{2}+5 A_{1}^{2} A_{2}^{2}\right)+A_{1}^{10}+A_{2}^{10} \\
12 D= & \left(A_{1}^{5}-A_{2}^{5}\right)\left(-1024 A_{0}^{10}+3840 A_{0}^{8} A_{1} A_{1}\right. \\
& -3840 A_{0}^{5} A_{1}^{2} A_{2}^{2}+1200 A_{0}^{4} A_{1}^{3} A_{2}^{3} \\
+ & \left.-100 A_{0}^{2} A_{1}^{4} A_{2}^{4}+A_{1}^{5} A_{2}^{5}\right) \\
+ & \left(A_{1}^{10}-A_{2}^{10}\right)\left(352 A_{0}^{4}-160 A_{0}^{2} A_{1}^{15}\right)
\end{aligned}
$$

According to Klein the ring of I-invariant polynomials is given as follows
(14)

$$
\mathbb{C}\left[A_{0}, A_{1}, A_{1}\right]^{I}=C[A, B, C, D] /(R(A, B, C, D)=0)
$$

The relation $R(A, B, C, D)=0$ is of degree 30 .
We have
(15)

$$
\begin{aligned}
& R(A, B, C, D)= \\
& -144 D^{2}-1728 B^{5}+720 A C B^{3}-80 A^{2} C^{2} B \\
& +64 A^{3}\left(5 B^{2}-A C\right)^{2}+C^{3} .
\end{aligned}
$$

The equations for $B$ and $C$ show that the two tangents of $B=0$ in the pole $(\sqrt{5} / 2,0,0)$ are given by $A_{1}=0, A_{2}=0$. They coincide with the tangents of $C=0$ in that pole. Therefore the curves $B=0$ and $C=0$ have in each pole the intersection multiplicity 10. Thus they intersect only in the poles.

When we restrict the action of $I$ to the conic $A=0$, we get the wellknown action of $I$ on $P_{1}(\mathbb{C})$ (which can also be obtained via the isomorphism $S O(3) \cong \operatorname{PSU}(2))$. The curves $B=0, C=0, D=0$ intersect $A=0$ tranversally in $12,20,30$ points respectively. If one uses a suitable conformal map $S^{2} \rightarrow P_{1}(\mathbb{C}) \cong\{A=0\}$ these points correspond to the 12 vertices, 20 center points of the faces, 30 center points of the edges of the icosahedron (always projected from the origin of $\mathbb{R}^{3}$ to $S^{2}$ ). Putting $A=0$, the relation $R(A, B, C, D)=0$ gives a famous icosahedral identity.

We consider the uniquely determined double cover $W$ of $P_{2}(\mathbb{C})$ branched along $C=0$. The action of $I$ can be lifted to the double cover.

The study of the Hilbert modular surface $H^{2} / \Gamma$ Ied to an action of $G / \Gamma\left(\cong A_{5}\right)$ on the complex projective plane. We also found the invariant curve $C=0$. We use an isomorphism $G / \Gamma \cong I$ to identify $G / \Gamma$ and the icosahedral group. Since the action of $I$ on the projective plane is essentially unique and the invariant curve $C=0$ well detemined as curve of degree 10 with double cusps in the poles, we have proved the following result.

THFOREM. Let $\Gamma$ be the principal congruence subgroup of $S_{2}(0)$ for the ideal $(\sqrt{5})$ in the ring 0 of integers of the field $\mathbb{Q}(\sqrt{5})$. Then the Hilbert modular surface $H^{2} / \Gamma$ can be compactified by six points (cusps in the sense of modular surfaces) to give a surface $\overline{H^{2} / \Gamma}$ with these cusps as the only singular points. The surface $\overline{H^{2} / \Gamma}$ admits an action
of the icosahedral group I. It is I-equivariantly isomorphic to the double cover $W$ of $P_{2}(\mathbb{L})$ branched along the Klein curve $C=0$. This curve has singularities ("double cusps") in the six poles of the action I and otherwise no singularities. The double cover $W$ has a singular point above each double cusp of $C$ and no further singular points. Under the isomorphism these singular points correspond to the six singular points of $\overline{H^{2} / \Gamma}$. The involution of the double cover $W$ corresponds to the involution of $\overline{\mathrm{H}^{2} / \Gamma}$ induced by $\left(z_{1}, z_{2}\right) \rightarrow\left(z_{2}, z_{1}\right)$ on $\mathrm{H}^{2}$. The surface $W$ is rational.

We use this theorem to gain information on the modular forms relative to $\Gamma$. A modular form of weight $k$ is a holomorphic function $f\left(z_{1}, z_{2}\right)$ on $H^{2}$ transforming under elements of $\Gamma$ as recalled in $\S 2$. The form $f$ is a cusp form if it vanishes in the cusps. The forms of weight 2 r correspond bijectively to the holomorphic sections of $K^{r}$ where $K$ is the canonical bundle of $\mathrm{H}^{2} / \Gamma$. A form is symmetric if $f\left(z_{1}, z_{2}\right)=f\left(z_{2}, z_{1}\right)$, skew-symmetric if $f\left(z_{1}, z_{2}\right)=-f\left(z_{2}, z_{1}\right)$. Let $W$ ' be the double cover $W$ of $P_{2}(\mathbb{C})$ with the six singular points removed and $P_{2}^{\prime}(\mathbb{Q})$ the projective plane with the six poles removed. Let $\pi: W^{\prime} \rightarrow P_{2}^{\prime}(\mathbb{C})$ be the covering map, $\tilde{\gamma}$ the divison in $W^{\prime}$ represented by the branching locus $C=0$ and $\gamma$ the divisor in $P_{2}^{\prime}(\mathbb{C})$ given by $C=0$. If $L$ is a line in $P_{2}^{\prime}(\mathbb{C})$, then $\tilde{\gamma}+\pi^{*}(-3 L)$ is a canonical divisor of $W^{\prime}$. Because $\pi^{*} \gamma=2 \tilde{\gamma}$, we conclude that $\pi^{*}(\gamma-6 L)$ is twice a canonical divisor and also $\pi^{*}(4 \mathrm{~L})$ is twice a canonical divisor on $W^{\prime}$. Therefore, under the isomorphism $H^{2} / \Gamma \rightarrow W^{\prime}$, a homogeneous polynomial of degree 4 r in $A_{0}, A_{1}, A_{2}$ defines a section of $K^{2 r}$ and thus a modular form relative to $\Gamma$ of weight 4 r . It can be proved, that the abelian group $\Gamma /[\Gamma, \Gamma]$ has a trivial 2 -primary component. This implies that a homogeneous polynomial of degree $k$ in $A_{0}, A_{1}, A_{2}$ defines a modular form relative to $\Gamma$ of weight $k$. In fact, these modular forms are symmetric. There is a skew-symmetric form of weight 5 , whose divisor is $\tilde{\gamma}$ (under the
isomorphism $\left.H^{2} / \Gamma \rightarrow W^{\prime}\right)$. We denote it by $c$. Thus we have obtained a graded subring

$$
M^{\prime}(\Gamma)=\sum_{k \geq 0} M_{k}^{\prime}(\Gamma)=\mathbb{C}\left[A_{0}, A_{1}, A_{2}, c\right] /\left(c^{2}=C\right)
$$

of the full graded ring $M(\Gamma)=\sum M_{k}(\Gamma)$ of modular forms for $\Gamma$. (Here $C$ is the Klein polynomial of degree 10.) We have

$$
\begin{aligned}
& \operatorname{dim} M_{k}^{\prime}(\Gamma)=\binom{k+2}{2}+\binom{k-3}{2}=k^{2}-2 k+7 \quad \text { for } k \geqq 3 \\
& \operatorname{dim} M_{2}^{\prime}(\Gamma)=6 \\
& \operatorname{dim} M_{1}^{\prime}(\Gamma)=3
\end{aligned}
$$

The arithmetic genus $X$ of the non-singular model $Y$ of $\overline{H^{2} / \Gamma}$ equals 1 , because $Y$ is rational. The dimension formula ( $\$ 2$ (4)) and $\$ 3$ (5) imply that $M_{k}^{\prime}(\Gamma)=M_{k}(\Gamma)$ for $k \geqq 2$. We have to use that there exist six Eisenstein series of weight $k$ (for $k \geq 2$ ) belonging to the six cusps which shows $\operatorname{dim} M_{k}(\Gamma)-\operatorname{dim} S_{k}(\Gamma)=6$ for $k \geqq 2$. Because the square of a modular form $f$ of weight one belongs to $M_{2}(\Gamma)=M_{2}^{\prime}(\Gamma)$, the zero divisor of $f$ gives a line in $P_{2}(\mathbb{Q})$. Thus $M_{1}(\Gamma)=M_{1}^{\prime}(\Gamma)$. of course, there are no modular forms of negative weight.

THEOREM. The ring of modular forms for the group $\Gamma$ is isomorphic to

$$
\mathbb{a}\left[A_{0}, A_{1}, A_{2}, c\right] /\left(c^{2}=c\right)
$$

The ring of symmetric modular forms for $\Gamma$ is

$$
\mathbb{C}\left[A_{0}, A_{1}, A_{2}\right]
$$

The vector space of skew-symmetric forms is

$$
c \cdot \mathbb{C}\left[A_{0}, A_{1}, A_{2}\right]
$$

The group $G / \Gamma=I=$ icosahedral group operates on these spaces by the

Klein representation of I of degree 3 in terms of the coordinates $A_{0}, A_{1}, A_{2}$ of $\mathbb{C}^{3}$.

We now consider the full Hilbert modular group $G=S L_{2}(0) /\{ \pm 1\}$ for $Q(\sqrt{5})$ and obtain in view of (14) and (15).

THEOREM. The ring of modular forms for the group $G$ is isomorphic to

$$
\begin{aligned}
\mathbb{C}[A, B, C, D] /\left(144 D^{2}\right. & =-1728 B^{5}+720 A C^{2} B^{3}-80 A^{2} C^{4} B \\
& \left.+64 A^{3}\left(5 B^{2}-A C^{2}\right)^{2}+C^{6}\right)
\end{aligned}
$$

The ring of symmetric modular forms for $G$ is isomorphic to

$$
\begin{equation*}
\mathbb{C}[A, B, C, D] /(R(A, B, C, D)=0) \tag{16}
\end{equation*}
$$

For the preceding theorems compare the papers of Gundlach [3] and Resnikoff [16] and also [8] where results on $\mathbb{Q}(\sqrt{5})$ where derived using the principal congruence subgroup of $\mathrm{SL}_{2}(\mathrm{O})$ for the prime ideal (2). In [8] the relation $R(A, B, C, D)=0$ was obtained in a different form connected to the discriminant of a polynomial of degree 5. The modular form D occurs in Grundlach's paper [3] as a product of 15 modular forms for $\Gamma$ of weight 1 each cuspidal at 2 cusps and vanishing along the "line" between these 2 cusps. The zero divisors of the six Eisenstein series for $\Gamma$ of weight 2 correspond to the six conics passing through 5 of the six poles. (Each Eisenstein series is cuspidal in five cusps.) In $H^{2} / G$ the curve $C=0$ becomses $F_{1}$ (given by $z_{1}=z_{2}$ ). The restriction of $B$ to $F_{1}$ gives a cusp form of weight 12 on $\mathrm{H} / \mathrm{SL}_{2}(\mathbb{Z})$, therefore must be $\Delta$ (up to a factor). The curves $\mathrm{B}=0$, $C=0$ intersect only in the six poles of the action of $I$, in agreement with the fact that $\Delta$ does not vanish on $H$.

Remark. I. Naruki has given a geometric interpretation of the curve
$B=0$. Let $S(5)$ be the elliptic modular surface in the sense of T. Shioda associated to the principal congruence subgroup $\Gamma(5)$ of $S L_{2}(\mathbb{Z})$. Choose a "zero section" $\sigma$ of $S(5)$, then each regular fibre of $S(5)$ becomes a group (1-dim. complex torus). The binary icosahedral group $I^{\prime}=S L_{2}\left(\mathbb{F}_{5}\right)$ is the group of automorphisms of $S(5)$ which carry o to itself. The element $-1 \in I^{\prime}$ acts as the involution which is $x \rightarrow-x$ on each regular fibre. Dividing $S(5)$ by this involution and blowing down 24 exceptional curves which come from the 12 singular fibres of $S(5)$ gives $P_{1}(\mathbb{C}) \times P_{1}(\mathbb{C})$ on which $I=I^{\prime} /\{ \pm 1\}$ operates. Dividing $P_{1}(\mathbb{C}) \times P_{1}(\mathbb{C})$ by the natural involution interchanging components yields $P_{2}(\mathbb{C})$ on which $I$ acts by the Klein representation. Under this procedure $B=0$ is the image of the curve in $S(5)$ containing all the points of the regular fibres of $S(5)$ which have precisely the order 4. A paper of Naruki (über die Kleinsche Ikosaeder-Kurve sechsten grades) will appear in Mathematische Annalen.
§4. THE FIELD $K=\mathbb{Q}(\sqrt{2})$.

In this field the ring 0 of integers consists of all linear combinations $a+b \sqrt{2}$ with $a, b \in \mathbb{Z}$. The fundamental unit is $\varepsilon_{0}=1+\sqrt{2}$. We consider the principal subgroup $\widetilde{\Gamma}(2)$ of $S L_{2}(0)$ for the ideal (2). The group $\widetilde{\Gamma}(2) /\{ \pm 1\}$ is a subgroup $\Gamma(2)$ of the Hilbert modular group $G=S L_{2}(0) /\{ \pm 1\}$. The group $G / \Gamma(2)$ is an extension of the symmetric group $S_{4}$ by a group of order 2 (which is the center of $G / \Gamma(2)$ ). The non-trivial element in the center is represented by the matrix

$$
\left(\begin{array}{ll}
\varepsilon_{0} & 0 \\
0 & \varepsilon_{0}^{-1}
\end{array}\right)=D \varepsilon_{0}
$$

of $\mathrm{SL}_{2}(0)$. Let F be the subgroup of $G$ obtained by extending $\Gamma(2)$ by
$D_{\varepsilon_{0}}$. Then $G / \Gamma \cong S_{4}$. The group $\Gamma$ acts freely on $H^{2}$. We shall investigate $\Gamma$ similarly as we treated the congruence subgroup with respect to $(\sqrt{5})$ in $\S 3$. Often details will be omitted an proofs only skecthed.

The Hilbert modular surface $\overline{H^{2} / \Gamma(2)}$ has six cusps, each resolved by a cycle of type $(4,2,4,2)$ ). The non-singular surface thus obtained will be called $Y_{2}$. The curve $F_{1}$ in $\overline{H^{2} / G}$ is given by $z_{1}=z_{2}$, the curve $F_{2}$ by $\lambda z_{2}-\lambda{ }^{\prime} z_{1}=0$ with $\lambda=\sqrt{2} \cdot \varepsilon_{0}$. The inverse images of $F_{1}$ and $F_{2}$ in $Y_{2}$ are also denoted by $F_{1}$ and $F_{2}$ respectively. $F_{1}$ has 8 and $F_{2}$ has 6 components in $Y_{2}$. The curves $F_{1}$ and $F_{2}$ in $Y_{2}$ pass through each of the six resolved cusps as follows
(17)


The 14 components of $F_{1} \cup F_{2}$ are disjoint, non-singular rational curves. Each component of $F_{1}$ has selfintersection number -1 , hence is an
exceptional curve. Each component of $F_{2}$ has selfintersection number -2 . Because $2 \zeta_{K}(-1)=\frac{1}{6}$, the Euler number of $H^{2} / \Gamma(2)$ is $48 / 6=8$, and we have (as in $\$ 3$ (8))

$$
e\left(Y_{2}\right)=8+6.4=32
$$

In fact, $Y_{2}$ is a $K 3$-surface with 8 points blown up. This can be shown by the methods of [9], see [2]. The involution on $Y_{2}$ given by $D_{\varepsilon_{0}}$ will be denoted by $\delta$. It operates freely on $Y_{2}$. The non-singular model $Y$ for $\overline{H^{2} / \Gamma}$ (obtained by resolving the six cusps) equals $Y_{2} / \delta$. Therefore, $Y$ has Euler number 16 , it is an Enriques surface with 4 points blown up. (An Enrique surface can be defined as a surface with fundamental group of order 2 whose universal covering is a k3-surface.) Each cusp of $\overline{H^{2} / \Gamma}$ is resolved by a cycle of type ( 4,2 ) (type ( $\left.4,2,4,2\right)$ ) divided by $\delta$ ). The inverse inage of $F_{1}$ and $F_{2}$ in $Y$ are also called $F_{1}, F_{2}$. They have 4 or 3 components respectively, the four components of $F_{1}$ being exceptional curves. The curves $F_{1}$ and $F_{2}$ in $Y$ pass through each of the six resolved cusps as follows
(18)


The involution $\tau:\left(z_{1}, z_{2}\right) \rightarrow\left(z_{2}, z_{1}\right)$ on $H^{2}$ induces an involution $\tau$ on $Y$, because $A \in \Gamma \leftrightarrow A^{\prime} \in \Gamma$. It commutes with the action of every element of $G / \Gamma \cong S_{4}$ on $Y$, because $A, A$ are equivalent mod 2 . The fixed point set of $\tau$ on $Y$ is $F_{1} \cup F_{2}$. We have $e\left(F_{1}\right)=8$ and $e\left(F_{2}\right)=6$.

Therefore

$$
e(Y / \tau)=\frac{1}{2}\left(e(Y)+e\left(F_{1}\right)+e\left(F_{2}\right)\right)=\frac{1}{2}(16+8+5)=15
$$

We now look at example b) of $\S 1$ and see that, from each cusp, $Y / \tau$ has 2 curves to blow down successively. We blow down these 12 curves and obtain a surface $Y_{0}$ with e $\left(Y_{0}\right)=3$. If $A$ is a component of $F_{1}$ on $Y_{0}$ and $B$ a component of $F_{2}$ on $Y_{0}$, then a simple calculation shows $c_{1}(A)=3$ and $c_{1}(B)=6$ where $c_{1}$ is the first Chern class of $Y_{0}$. Therefore $Y_{0}$ is rational and is in fact the projective plane $P_{2}(\mathbb{C})$, on which $F_{1}$ becomes a union of 4 lines intersecting in 6 points and $F_{2}$ a union of 3 conics with a contact point in each of the six points (compare example $b$ ) in 51 ). The group $G / \Gamma \cong S_{4}$ operates on $Y_{0}=P_{2}(\mathbb{C})$ with $F_{1} \cup F_{2}$ as an invariant curve of degree 10 . The isomorphism $G / \Gamma \cong S_{4}$ is established by the permutation of the four components of $F_{1}$. There is up to projective equivalence only one projective representation of $S_{4}$ permuting four lines in general position. It can be lifted in 2 ways to a linear representation :
Embed $\mathbb{a}^{3}$ in $\mathbb{a}^{4}$ by

$$
\begin{equation*}
x_{1}+x_{2}+x_{3}+x_{4}=0 \tag{19}
\end{equation*}
$$

Take the permutations of $x_{1}, x_{2}, x_{3}, x_{4}$ (representation $p_{1}$ of $S_{4}$ ) or the permutations followed by multiplication with their signs (representation $\rho_{1}$ of $S_{4}$ ).

Consider the projective plane with homogeneous coordinates $x_{1}, x_{2}, x_{3}, x_{4}$ subject to (19). Then

$$
F_{1} \text { is given by } x_{1} \cdot x_{2} \cdot x_{3} \cdot x_{4}=0
$$

because this is the only invariant curve of degree 4 which has 4 lines as components. The six cusps of $\overline{H / T}$ correspond to the six intersection points $(0,0,1,-1)$ (and permutations) of the 4 lines. Furthermore,

$$
F_{2} \text { is given by }\left(x_{1} x_{2}+x_{3} x_{4}\right)\left(x_{1} x_{3}+x_{2} x_{4}\right)\left(x_{1} x_{4}+x_{2} x_{3}\right)=0
$$

because this is the only invariant curve of degree 6 passing through $(0,0,1,-1)$ with 3 irreducible conics as components. Let $\sigma_{k}$ be the $k^{\text {th }}$ elementary symmetric function of $x_{1}, x_{2}, x_{3}, x_{4}\left(\sigma_{1}=0\right)$. The polynomial

$$
\begin{align*}
c & =x_{1} x_{2} x_{3} x_{4}\left(x_{1} x_{2}+x_{3} x_{4}\right)\left(x_{1} x_{3}+x_{2} x_{4}\right)\left(x_{1} x_{4}+x_{2} x_{3}\right)  \tag{20}\\
& =o_{4}\left(\sigma_{3}^{2}-4 \sigma_{2} \sigma_{4}\right)
\end{align*}
$$

of degree 10 describes the branch locus $F_{1} \cup F_{2}$.

THEOREM. Let $\Gamma$ be the extended principal congruence subgroup of $G=S_{2}(0) /\{ \pm 1\}$ for the ideal (2) in the ring of integers of the field $Q(\sqrt{2})$. Then $\overline{H^{2} / \Gamma}$ is isomorphic to the double cover $W$ of $P_{2}(\mathbb{C})$ along the curve $C=0$ of degree 10 . This curve has exactly 6 singular points which give singular points of $W$ corresponding to the six cusps of $\overline{\mathrm{H}^{2} / \Gamma}$. Desingularizing $W$ in the canonical way gives a surface $Y$ which is an Enriques surface with 4 points blown up. (The exceptional points in $Y$ come from the 4 Iinear components of $C=0$.

To gain information for the modular forms relative to $\Gamma$, one has to deal with difficulties arising from the fact that $\Gamma$ has a non-trivial character $\Gamma \rightarrow\{1,-1\}$. If one compares with the result of Gundlach [4] where these "sign questions" were treated, one can prove as in $\$ 3$ that the ring of modular forms for the group $\Gamma$ is isormorphic to

$$
\begin{equation*}
\mathbb{C}\left[x_{1}, x_{2}, x_{3}, x_{4}, c\right] /\left(a_{1}=0, c^{2}=c\right) \tag{21}
\end{equation*}
$$

This checks with the dimension formula (\$2 (4)), because as in §3 we have $e\left(H^{2} / \Gamma\right)=4$ and $X=1$ (since $Y$ is an Enriques surface). Comparing with Gundach [4] shows in addition that $G / \Gamma \cong S_{4}$ operates on the ring (21) by the representation $\rho_{2}$. The ring of invariant polynomials
for this representation is generated by $\sigma_{2}, \sigma_{4}, \sigma_{3}^{2}, \sigma_{3} \Delta$ where $\Delta=\Pi_{i<i}\left(x_{i}-x_{j}\right)$ is the discriminant. We have a relation $R\left(\sigma_{2}, \sigma_{4}, \sigma_{3}^{2}, \sigma_{3} \Delta\right)=0$ for these generators, namely

$$
\begin{align*}
& R\left(\sigma_{2}, \sigma_{4}, \sigma_{3}^{2}, \sigma_{3} \Delta\right)=27\left(\sigma_{3} \Delta\right)^{2}+  \tag{22}\\
& {\left[-4\left(\sigma_{2}^{2}+12 \sigma_{4}\right)^{3}+\left(27 \sigma_{3}^{2}+2 \sigma_{2}^{3}-72 \sigma_{2} \sigma_{4}\right)^{2}\right] \sigma_{3}^{2}}
\end{align*}
$$

which can be taken from the formula for the discriminant of a polynomial of degree 4. It follows

THEOREM. The ring of symmetric modular forms for the Hilbert modular group $G=S L_{2}(0) /\{ \pm 1\}$ of the field $\mathbb{Q}(\sqrt{2})$ is isomorphic to

$$
\mathbb{C}\left[\sigma_{2}, \sigma_{4}, \sigma_{3}^{2}, \sigma_{3} \Delta\right] /\left(R\left(\sigma_{2}, \sigma_{4}, \sigma_{3}^{2}, \sigma_{3} \Delta\right)=0\right)
$$

This agrees with Gundlach [4], Satz 1. But there the relation was not determined. The ring of modular forms for $G$ is obtained attaching the skew-symmetric form $c$ of weight 5 satisfying

$$
c^{2}=c=\sigma_{4}\left(\sigma_{3}^{2}-4 \sigma_{2} \sigma_{4}\right)
$$

The modular forms $G, \tilde{H}, H, \theta$ (belonging to various characters of $\left.S L_{2}(0) /\{ \pm 1\}\right)$ which Gundlach [4] mentions in his Theorem 1 find the following description in our set up (up to a factor). We also give the zero divisors.

$$
\begin{aligned}
& G=\Delta \quad \text {, (six lines) } \\
& \widetilde{H}=\sigma_{3} \text {, (three lines) } \\
& H=\sqrt{\sigma}_{3}^{2}-4 \sigma_{2} \sigma_{4} \text {, (part of the branching locus; three } \\
& \theta=\sqrt{0}_{4} \quad \text {, conics) } \text { (part of the branching locus; four } \\
& \text { lines) }
\end{aligned}
$$

The theory we have developed for $\mathbb{Q}(\sqrt{2})$ involves the symmetry group $S_{4}$ of a cube. Similar considerations for $\phi(\sqrt{3})$ are possible, but more complicated. Here the group $A_{4}$ (symmetry group of a tetrahedron) enters. Gundlach [4] has also investigated $Q(\sqrt{3})$, but the translation into our geometric method must be done at some other occasion.
§5. ON THE FIELDS $\mathbb{Q}(\sqrt{7})$ AND $\mathbb{C}(\sqrt{13})$.

In $\mathbb{Q}(\sqrt{7})$ there is no unit of negative norm. Therefore, we consider the extended group $\mathrm{GL}_{2}^{+}(0)$ of all matrices $\left(\begin{array}{ll}\alpha & \beta \\ \gamma & \delta\end{array}\right)$ with $\alpha, \beta, \gamma, \delta \in 0$ and determinant a totally positive unit. For the prime ideal ( $\sqrt{7}$ ) let $\Gamma^{+}(\sqrt{7})$ consist of all matrices of $\mathrm{GL}_{2}^{+}(0)$ which are congruent to $\pm\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right) \bmod (\sqrt{7})$. Let $D$ be the group of diagonal matrices $\left(\begin{array}{ll}\varepsilon & 0 \\ 0 & \varepsilon\end{array}\right)$ with $\varepsilon$ a unit. Since the fundamental unit $\varepsilon_{0}$ equals $8+3 \sqrt{7}$, this diagonal group is contained in $\Gamma^{+}(\sqrt{7})$. The groups $G L_{2}^{+}(0) / D$ and $\Gamma^{+}(\sqrt{7}) / D$ operate effectively on $H^{2}$. We denote them by $G^{+}$and $F$ respectively. $G^{+}$is the extended Hilbert modular group with $\left[G^{+}: G\right]=2$ where $G=S L_{2}(0) /\{ \pm 1\}$. We have

$$
\mathrm{G}^{+} / \Gamma \cong \mathrm{PSL}_{2}\left(\mathbb{F}_{7}\right)=\mathrm{G}_{168}
$$

This is the famous simple group of order 168. The group $I$ operates freely on $H^{2}$. The surface $\mathrm{H}^{2} / \Gamma$ is compactified by 24 points (cusps). Each cusp is resolved as in 51 (example c). This gives a non-singular surface $Y$. Because $\zeta_{\mathbb{O}(\sqrt{7})}(-1)=\frac{2}{3}$, we have

$$
e(Y)=\frac{2}{3} .168+5.24=232
$$

We consider the curves $F_{1}, F_{2}, F_{4}$ in $\overline{H^{2} / G^{+}}$. They are given by $z_{1}=z_{2}$, $(3+\sqrt{7}) z_{2}-(3-\sqrt{7}) z_{1}=0$ and $z_{1}-z_{2}=\sqrt{7}$ respectively. Their inverse images in $Y$ will also be denoted by $F_{1}, F_{2}, F_{4}$. These are non-singular disjoint curves in $Y$. They pass through each of the 24 cusps as

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follows
(23)


The Euler numbers of $F_{1}, F_{4}, F_{2}$ in $Y$ are given by

$$
\begin{aligned}
& e\left(F_{1}\right)=-\frac{1}{6} \cdot 168+24=-4 \\
& e\left(F_{4}\right)=-\frac{1}{4} \cdot 168+24=-18 \\
& e\left(F_{2}\right)=-\frac{1}{4} \cdot 168+24=-18
\end{aligned}
$$

because $-\frac{1}{6},-\frac{1}{4},-\frac{1}{4}$ are the normalized Euler volumes of the curves $F_{1}, F_{4}, F_{2}$ in $H^{2} / G^{+}$.
The involution $\left(z_{1}, z_{2}\right) \rightarrow\left(z_{2}, z_{1}\right)$ of $H^{2}$ induces an involution $\tau$ of $Y$ commuting with each element of $G^{+} / \Gamma \cong G_{168}$. The fixed point set of $\tau$ in $Y$ is $F_{1} \cup F_{4} \cup F_{2}$. Therefore,
(24)

$$
e(Y / \tau)=\frac{1}{2}(232-4-18-18)=96
$$

The example c) in $\$ 1$ shows that each cusp gives rise to three curves which can be blown down successively. We obtain a surface $Y_{0}$ with

$$
e\left(Y_{0}\right)=96-3.24=24
$$

The group $G_{168}$ acts on $Y_{0}$. One can proof that $Y_{0}$ is rational. There is a famous action of $G_{168}$ on $P_{2}(\mathbb{Q})$, see [18], §88, §133-140. This action has an orbit consisting of 21 points. Up to an equivariant isomorphism
$Y_{0}$ is obtained from $P_{2}(\mathbb{C})$ by blowing up these 21 points. The curves $F_{1}, F_{4}, F_{2}$ become invariant curves of degrees $4,18,12$.
This result has to be proved in some other paper. It should be used to investigate the structure of the ring of Hilbert modular forms relative to $\Gamma$ and $G^{+}$.

Our last example concerns the field $p(\sqrt{13})$. It is due to van der Geer [2] who has proved many interesting results on the Hilbert modular surfaces of principal congruence subgroups. Let 0 be the ring of integers in $Q(\sqrt{13})$. Let $\tilde{\Gamma}$ be the congruence subgroup of $S_{2}(0)$ for the prime ideal 2 of 0 . Then $\Gamma=\widetilde{\Gamma} /\{ \pm 1\}$ is a normal subgroup of $G=S L_{2}(0) /\{ \pm 1\}$. The quotient group is $S L_{2}\left(\mathbb{F}_{4}\right) \cong A_{5}$. We consider the Hilbert modular surface $\overline{H^{2} / \Gamma}$. It has 5 cusps. Each is resolved as in $\S 1$, example d). Let $Y$ be the non-singular surface obtained in this way. Since

$$
{ }_{\Phi}^{2 \zeta(\sqrt{13})}(-1)=\frac{1}{3}
$$

we have

$$
e(Y)=\frac{1}{3} \cdot 60+5.9=65
$$

The inverse image in $Y$ of the curve $F_{1}$ on $\overline{H^{2} / G}$ has 10 disjoint components which are non-singular rational curves of selfintersection number -1. (Proof as in [8]). The inverse image will also be denoted by $F_{1}$. It passes through each of the five cusps as follows
(25)


Each component of $E_{1}$ goes through 3 of the 5 cusps and is determined by these three cusps. We blow down the ten components of $F_{1}$ and obtain a surface $Y_{1}$ of Euler number 55. It has arithmetic genus $5=\frac{1}{4} e\left(H^{2} / \Gamma\right)$, see [7]. Therefore $P_{g}=4$. The surface $Y_{1}$ is a minimal surface of general type. The space of sections of the canonical bundle $K$ of $Y_{1}$ is isomorphic to the space of cusp forms $S_{2}(\Gamma)$. The cusp forms define a "map"

$$
\phi_{K}: Y_{1} \rightarrow P_{3}(\mathbb{C})
$$

The action of $G / \Gamma \cong A_{5}$ on $S_{2}(\Gamma)$ is the standard action on $\mathbb{C}^{4}$ represented in $\mathbb{T}^{5}$ by

$$
x_{1}+x_{2}+x_{3}+x_{4}+x_{5}=0
$$

It turns out that $\phi_{K}$ is holomorphic of degree 1 and $\phi_{K}\left(Y_{1}\right)$ is given in $P_{4}(\mathbb{C})$ Dy

$$
\begin{equation*}
\sigma_{1}=0, \quad \sigma_{2} \sigma_{3}-2 \sigma_{5}=0 \tag{26}
\end{equation*}
$$

where $\sigma_{k}$ is the $k^{\text {th }}$ elementary symmetric function of $x_{1}, \ldots, x_{5}$. The surface (25) has 15 double points which are images under $\phi_{K}$ of the 15 configurations $\longrightarrow-2<-2$ on $Y_{1}$ (see (25)). Otherwise $\phi_{K}$ is bijective. Because (26) gives a relation between the cusp forms of weight 2, it can be used to gain more information on the ring of modular forms for $\Gamma$ (see [2]).

The ideal (2) does not divide the discriminant of $\mathbb{Q}(\sqrt{13})$. Therefore, we do not have an involution $\tau$ on $Y$ commuting with $G / \Gamma$.

## Remarks.

1) The surface $Y_{1}$ is diffeomorphic to the general quintic hypersurface in $P_{3}(\mathbb{C})$.
2) Consider a subgroup of $A_{5}$ of order 5. It operates freely on $Y_{1}$. The quotient is a minimal surface of general type with arithmetic genus 1, Euler number 11 and Chern number $c_{1}^{2}=1$. We recall that

Godeaux has studied free actions of groups of order 5 on quintic surfaces and considered the corresponding quotients (L. Godeaux, Les surfaces algébriques non rationelles de genres arithmétique et géometrique nuls, Paris 1934).

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