CHERN CHARACTERISTIC CLASSES IN TOPOLOGY AND ALGEBRAIC GEOMETRY

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These are notes of my Oberwolfach lecture on October 10, 2009 (§9 was not treated in the lecture). It was a great pleasure and honor for me to give this talk. I thank the director, Professor Gert-Martin Greuel, very much for the invitation. I wish the MFO, its committees, the director and all the staff continued great success in future years.

§1. Introduction

I learnt about characteristic classes for the first time when I visited Heinz Hopf in Zurich in 1948 for one week after three weeks of work on a Swiss farm [1]. Heinz Hopf visited Oberwolfach already in 1946. At the meeting of the German Mathematical Society in Danzig in 1925 [2] he had reported about his work including what is now called Poincaré-Hopf theorem:

Let X be a compact differentiable manifold of dimension $n \geq 1$ and v a continuous vector field which is different from zero in all but finitely many points, also called singularities. Then the number of singularities each counted with its proper multiplicity is independent of the vector field. It is always equal to the Euler number $e(X)$.

The multiplicity of a singularity x is equal to the degree of a map of the $(n-1)$-dimensional sphere (boundary of a small neighborhood of x) to itself. This is well defined also for non-orientable manifolds. For n odd the Euler number $e(X)$ vanishes. There exist vector fields without singularities.

I learnt from Heinz Hopf the theory of Stiefel-Whitney classes for compact differentiable manifolds. E. Stiefel was a student of Hopf, who proposed to Stiefel the problem:

Which manifolds of dimension n admit an m-field, i. e. m vector fields $v_1, \ldots, v_m$ which are linearly independent everywhere? (Compare [3], §1.5 and §2.9).

But for me the Chern classes, first introduced by S. S. Chern in 1946 [4] became more important. The book by N. Steenrod [5] of 1951 was revealing.

Consider a compact complex manifold X of complex dimension n, for example a projective algebraic manifold embedded in some complex projective space $\mathbb{P}_N(\mathbb{C})$.

An r-field is an r-tuple of vector fields on X which are complex linearly independent outside a cycle of dimension $2r-2$ which determines an element in the homology group $H_{2r-2}(X,\mathbb{Z})$. A basic fact is that such r-fields always exist and the homology class of the "cycle of singularities" does not depend on the r-field. This is intuitive and not precise. The cohomology class corresponding to the "cycle of singularities" by the Poincaré isomorphism is the Chern class

$$c_{n-r+1} \in H^{2(n-r+1)}(X,\mathbb{Z})$$

which can be defined directly and precisely by obstruction theory [5]. For $r = 1$ we have the Poincaré-Hopf theorem applied to complex manifolds. It follows

$$c_n[X] = e(X)$$

§2. Chern numbers

We denote the Chern classes of a compact complex manifold X of complex dimension n simply by

$$c_i = c_i(X) \in H^{2i}(X,\mathbb{Z})$$

The index i runs from 0 to n, where c_0 denotes the unit element of the commutative graded cohomology ring

$$H^{ev}(X,\mathbb{Z}) = \bigoplus_{i=0}^{n} H^{2i}(X,\mathbb{Z})$$

The notation ev indicates even dimensional cohomology. We have the total Chern class

$$c = 1 + c_1 + \cdots + c_n \in H^{ev}(X,\mathbb{Z})$$

For any partition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n we consider the number

$$c_{\lambda_1} c_{\lambda_2} \ldots c_{\lambda_k}[X], \qquad \lambda_1 + \lambda_2 + \ldots \lambda_k = n \tag{1}$$

(evaluation of a $2n$-dimensional cohomology class on the fundamental cycle of X).

The cup product of cohomology classes is used. For the corresponding homology classes intersection theory comes in. Any linear combination of numbers like (1) with rational coefficients depending only on the partition is also called Chern number.

As mentioned before, the Chern number $c_n[X]$ equals the Euler number, thus is a topological invariant. In the problem collection [6] I formulate the question:

Which Chern numbers are topological invariants of projective algebraic manifolds? (Problem 31).

Very recently D. Kotschick [7] proved:

A Chern number is a topological invariant of projective algebraic manifolds if and only if it is a multiple of the Euler number.

For Kotschick's proof the following is basic: There exist pairs of algebraic surfaces X, Y which are homeomorphic with reversal of orientation and have non-vanishing signatures. The signature is a topological invariant of *oriented* 4-dimensional manifolds. For an algebraic surface X it is a Chern number:

$$\text{signature} = \frac{c_1^2[X] - 2c_2[X]}{3}$$

The signature changes sign under reversal of orientation. This proves Kotschick's theorem for $n = 2$. The existence of pairs of surfaces (X, Y) uses deep results of Friedman ([8], [9]). In fact, X and Y are not diffeomorphic. Kotschick has also results concerning invariance of Chern numbers under diffeomorphisms and under orientation preserving diffeomorphisms.

Kotschick's work was motivated in part by an old result of Borel-Hirzebruch [10], II. §24(11), in a modified formulation by E. Calabi. Consider the projective tangent bundle of $\mathbb{P}_3(\mathbb{C})$ and the projective covariant tangent bundle of $\mathbb{P}_3(\mathbb{C})$. These two five-dimensional projective algebraic manifolds are diffeomorphic. They have as Chern numbers c_1^5 the integers 4500 and 4860 respectively.

§3. Chern classes of complex vector bundles

We study complex vector bundles (C^∞-differentiable) with fibre $\mathbb{C}^n$ and basis a compact oriented differentiable manifold X of even dimension $2m$ (for reasons of exposition). For such a vector bundle E Chern classes $c_i \in H^{2i}(X, \mathbb{Z})$ can be defined. In §1 and §2 the bundle E was always the tangent bundle of a compact complex manifold of complex dimension n. In our more general case the Chern class c_i can be intuitively characterized by a cycle of dimension $2m - 2i$ given by the singularities of an $(n - i + 1)$-tuple of sections of E. For us the case $m = n$ and $i = n$ will be very important. This case is close to the Poincaré-Hopf theorem:

Take a section of E with isolated singularities. Then its number is finite and equal to $c_n[X]$ if each singularity is counted with its proper multiplicity, the degree of a map between two $(2n - 1)$-dimensional spheres: $S_1^{2n-1} \to S_2^{2n-1}$.

The first sphere is the boundary of a small neighborhood of x in X, the second sphere is the boundary of a small neighborhood of the origin in the fibre of E over x.

For the Poincaré-Hopf theorem the two spheres were identical and hence orientation not needed to define a mapping degree.

The number $c_n[X]$ is independent of the section.

The total Chern class of E can be formally written as

$$(2) \quad c(E) = 1 + c_1 + \cdots + c_n = (1 + x_1)(1 + x_2) \ldots (1 + x_n) \in H^{ev}(X, \mathbb{Z})$$

where the x_i are two-dimensional integral cohomology classes in some extension of $H^{ev}(X, \mathbb{Z})$. For this we can take H^{ev} of the flag manifold bundle $\mathfrak{F}$ associated to E. If we lift E to $\mathfrak{F}$, then E becomes a direct sum of line bundles L_i, and x_i is the first Chern class of L_i. This can also be taken as definition of the Chern classes.

Every symmetric polynomial with integral coefficients of degree k in the x_i is a polynomial with integral coefficients in the elementary symmetric functions and hence a polynomial in the Chern classes c_i defining a cohomology class of dimension $2k$.

Formula (2) and these applications represent the splitting principle in [11].

Let E and F be vector bundles of dimensions r and s over the same base manifold X. Then we have for the total Chern classes

$$c(E \oplus F) = c(E)c(F),$$

in particular for the highest Chern class

$$c_{r+s}(E \oplus F = c_r(E)c_s(F).$$

This follows easily from the splitting principle. In the sequel we shall use freely the formula for $c(E \oplus F)$. We shall make a few intuitive remarks for the highest Chern class.

§4. The highest Chern class

The operations of multilinear algebra like direct sum, tensor product, symmetric powers can be applied to complex vector bundles over X. The resulting new vector bundles have Chern classes which can be calculated as polynomials in the Chern classes of the given bundles. Some of this was developed and used in [11]. Here we are interested in the highest Chern class, because it describes the locus of zeros of a section. Let E and F be complex vector bundles of dimension r and s over X. For the direct sum we have

$$c_{r+s}(E \oplus F) = c_r(E)c_r(F) \tag{3}$$

Intuitively this is clear: Suppose we have sections in E and F, then their sum vanishes in $E \oplus F$ if and only if both sections vanish. this is the intersection of loci of zeros, corresponding to the cup product in cohomology.

Let E and F be line bundles over X. (The fibre dimension of E and F is one.) Then the tensor product is again a line bundle. We have

$$c_1(E \otimes F) = c_1(E) + c_1(F) \tag{4}$$

Intuitively: Suppose we have sections in E and F. In local coordinates the tensor product of sections is multiplication of complex numbers and vanishes if and only if one of the sections vanish: Union of the loci of zeros of E and F corresponding to addition in cohomology.

§5. The highest Chern class of the r-th symmetric power of a two-dimensional complex vector bundle

Consider a two-dimensional complex vector bundle V and suppose it is a direct sum of line bundles L_1 and L_2. With respect to a local trivialisation we can introduce coordinates ξ, η, such that L_1 is given by $\eta = 0$ and L_2 by $\xi = 0$. Then ξ and η define a local basis for the dual bundle V^* and locally the elements of the r-th symmetric power $S^r V^*$ are the polynomials

$$\sum_{k=0}^{n} a_k \, \xi^k \, \eta^{r-k}$$

From this we conclude that

$$S^rV^* = \bigoplus_{k=0}^{r} L_1^{*k} L_2^{*(r-k)}$$

where L_1^*, L_2^* are the dual line bundles of L_1, L_2. Let x, y be the first Chern class of L_1^*, L_2^*, then the highest Chern class of S^rV^* is given by

$$c_{r+1}(S^rV^*) = \prod_{k=0}^{r}(kx + (r-k)y). \tag{5}$$

This is a polynomial in $c_1 = x + y$ and $c_2 = xy$, the Chern classes of V^*. By the splitting principle (5) holds for an arbitrary 2-dimensional vector bundle V with c_1, c_2 being the Chern classes of V^*. We give some examples of (5).

$$\begin{aligned} c_4(S^3V^*) &= 9c_2(2c_1^2 + c_2) \\ c_6(S^5V^*) &= 5^2c_2(24c_1^4 + 58c_1^2c_2 + 9c_2^2) \\ c_8(S^7V^*) &= 7^2c_2(720c_1^6 + 3708c_1^4c_2 + 3004c_1^2c_2^2 + 225c_2^3) \end{aligned} \tag{6}$$

To indicate a fast way of calculation, we observe that

$$(ax + by)(bx + ay) = abc_1^2 + (a-b)^2c_2$$

and hence

$$c_6(S^5V^*) = 5^2c_2(4c_1^2 + 9c_2)(6c_1^2 + c_2)$$

§6. The space of lines in a projective space

We consider the space X_n of projective lines in the complex projective space $\mathbb{P}_{n+1}(\mathbb{C})$. This is also the space of 2-dimensional complex linear subspaces of $\mathbb{C}^{n+2}$. Thus it is the Grassmannian $G(2, n+2)$ which is a projective algebraic manifold of dimension $2n$. We can celebrate the 125th anniversary of Schubert calculus by remembering the paper by Hermann Caesar Hannibal Schubert published in 1885 [12]. There he considers the Schubert cycle consisting of all lines in $\mathbb{P}_{n+1}$ intersecting a given $\mathbb{P}_{n-1}$-subspace of $\mathbb{P}_{n+1}$. This is a very ample divisor D in X_n corresponding to the positive generator $f_1 \in H^2(X_n, \mathbb{Z}) \simeq \mathbb{Z}$. Schubert proved [12], §5(6):

$$f_1^{2n}[X_n] = \frac{(2n)!}{n!(n+1)!} = C_n = \text{the } n^{th} \text{ Catalan number.} \tag{7}$$

The Catalan numbers were introduced by Euler in a letter to Goldbach in 1751. Formula (7) is a special case of [10](formula (9) in 24.10 Theorem).

Let us recall the proof for our case: The holomorphic Euler number $\chi(X_n, rD)$ where the divisor D corresponds to f_1 is a polynomial in r of degree $2n$. The first Chern class of X_n equals $(n+2)f_1$. By the Kodaira vanishing theorem the holomorphic Euler number $\chi(X_n, rD)$ equals for $r > -(n+2)$ the dimension of $H^0(X_n, rD)$, the vector space of meromorphic functions f on X_n whose divisor (f) satisfies

$$(f) + rD \geq 0,$$

i. e. the divisor $(f) + rD$ does not have poles. In particular, the polynomial $\chi(X, rD)$ vanishes for $-(n+2) < r < 0$ and has the value 1 for $r = 0$. In fact,

$$\chi(X_n, rD) = \frac{(r+n)(r+2)^2 \dots (r+n)^2(r+n+1)}{1 \cdot 2^2 \cdot n^2 \dots (n+1)} \tag{8}$$

This follows from the Riemann-Roch-Hirzebruch formula [11] and the theory of roots as explained in [10]. The RRH-formula implies that the highest coefficient of our polynomial equals

$$\frac{f_1^{2n}[X_n]}{(2n)!}$$

Formula (7) follows.

The Catalan numbers C_n are for $n \geq 0$:

$$C_n \ : \ 1, 1, 2, 5, 14, 42, 132, 429 \dots$$

In Schubert's language using intersection theory formula (7) means the following:

Take $2n$ projective subspaces in $\mathbb{P}_{n+1}$ of dimension $n-1$ in general position, then the number of lines in $\mathbb{P}_{n+1}$ intersecting each of theses subspaces equals C_n.

The manifold X_n can also be written as

(9) $$X_n = \frac{U(n+2)}{U(2) \times U(n)}$$

Over X_n we have the tautological complex vector bundles E and F of dimensions 2 and n. For the point x of X_n the fibre E_x over x is the 2-dimensional subspace of $\mathbb{C}^{n+2}$ defining x and F_x is the quotient of $\mathbb{C}^{n+2}$ by E_x. By (9) there is an isomorphism

$$E_x \oplus F_x \simeq \mathbb{C}^{n+2}$$

Therefore $E \oplus F$ is the trivial bundle. We denote the Chern classes of E by e_1, e_2, those of F by $f_1, f_2, \ldots$. Then

$$(1 + e_1 + e_2)(1 + f_1 + f_2 + \ldots) = 1$$

.

Consider the dual bundle E^* and its Chern classes c_1, c_2, formally written as

$$(1 + x)(1 + y) = 1 + c_1 + c_2 = 1 - e_1 + e_2$$

Then

$$(1 - x)(1 - y)(1 + f_2 + f_3 + \ldots) = 1$$

and

$$\begin{aligned} f_1 &= c_1 = x + y \\ f_r &= x^r + x^{r-1}y + \cdots + y^r \end{aligned}$$

In Schubert's language f_r corresponds by the Poincaré isomorphism to the cycle of all lines intersecting a given $\mathbb{P}_{n-r}$. For $r = n$ this is a projective space $\mathbb{P}_n$ which indeed has codimension n in X_n. The cohomology classes f_r vanish for $r > n$, clear by Schubert, for us because the Chern classes of F vanish for $r > n$.

§7. The lines on a smooth projective hypersurface in $\mathbb{P}_{n+1}$

A smooth hypersurface of degree d in $\mathbb{P}_{n+1}$ is given by a homogeneous polynomial of degree d in $(n+2)$-variables, the coordinates of $\mathbb{C}^{n+2}$. Hence it is an element of $S^d((\mathbb{C}^{n+2})^*)$. It defines a section in the vector bundle $S^d(E^*)$ over X_n, because a fibre of E^* is the dual vector space of a 2-dimensional linear subspace of $\mathbb{C}^{n+2}$. The lines on the hypersurface correspond to the zeros of this section, their locus is given by the highest Chern class $c_{d+1}(S^dE^*)$, hence has complex dimension $2n-d-1$. This will be a subvariety of X_n. But we do not try to make this more precise. If $d+1=2n$, the number of lines is finite. Assuming that every line has multiplicity 1, we have:

The number of lines on a hypersurface of degree $2n-1$ in $\mathbb{P}_{n+1}$ is finite and equals $c_{2n}(S^{2n-1}E^*)[X_n]$.

We have to use the Chern classes c_1, c_2 of E^* where $c_1 = f_1 = x+y$ and $c_2 = xy$. The class c_2 is represented by the subvariety X_{n-1} of X_n of complex codimension 2, and c_1 restricted to X_{n-a} is the positive generator of $H^2(X_{n-a}, \mathbb{Z})$. Therefore by (7) for $a+b=n$

$$c_2^a\, c_1^{2b}[X_n] = C_b, \tag{10}$$

the b^{th} Catalan number.

Any homogeneous symmetric polynomial of degree $2n$ in x and y is a polynomial in c_1, c_2 of complex dimension $2n$ and can be evaluated on the fundamental cycle of X_n. The vector space of these polynomials has a basis consisting of the elements $c_2^a\, c_1^{2b}$ with $a+b=n$. Using (10) we can carry out this evaluation for any such polynomial. For the number of lines on smooth hypersurfaces we use (5) and express these numbers by Catalan numbers. The first example results from (6).

The number of lines on a hypersurface of degree 3 in $\mathbb{P}_3$ equals

$$9(2C_1 + C_0) = 27.$$

The number of lines on a hypersurface of degree 5 in $\mathbb{P}_4$ equals

$$5^2(24C_2 + 58C_1 + 9C_0) = 2875 = 5^3 \cdot 23.$$

The number of lines on a hypersurface of degree 7 in $\mathbb{P}_5$ equals

$$7^2(720C_3 + 3708C_2 + 3004C_1 + 225C_0) = 698005 = 7^3 \cdot 2035.$$

Remark: The number of lines on a hypersurface occurred as a Chern number. We had to neglect, for example, the long history of the discovery and study of the 27 lines on a cubic surface. On my desk there is the classical model of a cubic surface of Clebsch and Klein defined over the reals. Here also the 27 lines are defined over the reals. Thus I can see them any time I wish. In this special case the lines have 10 triple points.

§8. A DIFFERENT WAY TO DETERMINE THE NUMBER OF LINES ON A HYPERSURFACE

In the preceding section we loved to work with Catalan numbers using the basis $c_2^a\, c_1^{2b}$ with $a + b = n$ for the homogeneous symmetric polynomials in x, y of degree $2n$. The basis

$$c_2^a\, f_{2b} \qquad \text{with} \qquad a + b = n$$

may be more convenient. The evaluation of $c_2^a\, f_{2b}$ on X_n equals the evaluation of the Chern class f_{2b} on X_{n-a} where f_{2b} is now the Chern class of the complementary tautological vector bundle over X_{n-a} with fibre $\mathbb{C}^{n-a}$. But $2b = n - a + b$. Hence $c_2^a\, f_{2b}[X_n] = 0$ except for $b = 0$. Then $f_0 = 1$ and c_2^n is the generator of $H^{2n}(X_n, \mathbb{Z})$: For $b = 0$, we have $c_2^a\, f_{2b}[X_n] = 1$. Now we have the following result:

Let $P(x, y)$ be a symmetric polynomial in x, y of degree $2n$, then $P(x, y)[X_n]$ = coefficient of $x^{n+1}\, y^n$ in $(x - y)P(x, y)$.

This is clear because

$$(x - y)c_2^a\, f_{2b} = x^{2b+a+1}\, y^a - x^a\, y^{2b+a+1}$$

It follows:

The number of lines on a surface of degree $2n - 1$ in $\mathbb{P}_{n+1}$ equals the coefficient of $x^{n+1}\, y^n$ in $(x - y) \prod_{k=0}^{2n-1} (kx + (2n - 1 - k)y)$

This is van der Waerden's original theorem [13]. This paper has results about the configuration of lines which we cannot mention here.

The calculation with Catalan numbers is equivalent to van der Waerden's method, because the coefficient of $x^{n+1}y^n$ in $(x-y)(x+y)^{2n}$ equals $\binom{2n}{n} - \binom{2n}{n-1} = C_n$.

For the preparation of my Oberwolfach lecture I had discussions with Pieter Moree, who gave me the paper [14]. There I found the following asymptotic formula by Don Zagier:

Let v_n be the number of lines on a hypersurface of degree $2n-3$ in $\mathbb{P}_n$, then

$$v_n \sim \sqrt{\frac{27}{\pi}}(2n-3)^{2n-\frac{7}{2}}(1 - \frac{9}{8n} - \frac{111}{640n^2} - \frac{9999}{25600n^3} + \dots)$$

Don, the Oberwolfach lecturer of the preceding year, was present. I joked that he is unable to see a sequence of numbers without studying its asymptotic behavior.

§9. Higher dimensions

Consider the Grassmannian

$$X_{m,n} = \frac{U(m+n)}{U(m)\times U(n)} = G(m, m+n)$$

In the preceding sections we had $m=2$ and wrote $X_{2,n} = X_n$. Over $X_{m,n}$ we have the dual tautological vector bundle E^* with fibre $\mathbb{C}^m$ whose Chern classes generate the integral cohomology ring of $X_{m,n}$. We write the Chern classes of E^* as elementary symmetric functions in $x_1, x_2, \dots, x_m$. The Grassmannian $X_{m,n}$ has complex dimension mn. A generator of $H^{2mn}(X_m, \mathbb{Z}) \cong \mathbb{Z}$ with value 1 on the fundamental cycle is $(x_1 x_2 \dots x_m)^n$. In generalization of §8 we can prove:

Let $\mathbb{P}(x_1, \dots, x_m)$ be a symmetric polynomial in $x_1, \dots, x_m$ of degree mn, then

$$\mathbb{P}(x_1,\dots,x_m)\,[X_{m,n}] = \text{ coefficient of } x_1^{n+m-1}x_2^{n+m-2}\dots x_m^n \text{ in } \prod_{1\le i<j\le m}(x_i - x_j)\,\mathbb{P}(x_1,\dots,x_m) \tag{11}$$

In principle, we can use the preceding result to determine the number of $(m-1)$-dimensional projective subspaces on a hypersurface of degree d in $\mathbb{P}_{m+n-1}$. The number is finite if the fibre dimension of $S^d E^*$ (which is also the dimension of the highest Chern class of this vector bundle) equals the dimension of $X_{m,n}$. The condition is

$$(12) \qquad \binom{m-1+d}{m-1} = mn.$$

This is true for $m = 3, d = 4, n = 5$. A generic quartic in $\mathbb{P}_7$ contains $3297280 = 2^{12} \cdot 805$ projective planes. To get this number one has to evaluate
$64x_1x_2x_2(3x_1 + x_2)(3x_2 + x_3)(3x_1 + x_3)(x_1 + 3x_2)(x_2 + 3x_3)$
$(x_1 + 3x_3)(2x_1 + 2x_2)(2x_2 + 2x_3)(2x_1 + 2x_3)(x_1 + x_2 + 2x_3)$
$(x_1 + 2x_2 + x_3)(x_1 + x_2 + 2x_3)$
on $X_{3,5}$ using (11).

Another case which satisfies (12) is $m = 4$, $d = 3$, $n = 5$. A general cubic in $\mathbb{P}_8$ contains $1812768336 = 3^5 \cdot 7459952$ projective subspaces of dimension 3. The discussion in §9 up to this point I have from the book [15], in particular p. 132. A computer check-up for the case $m = 3$, $d = 4$, $n = 5$ was done for me by Chen Heng Huat (National University of Singapore, visitor of the MPI).

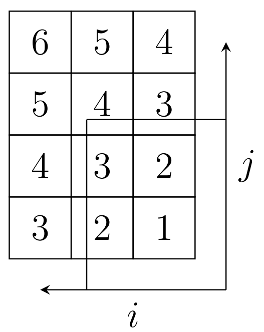
We would like to generalize the Catalan numbers, namely study the degree of the Grassmannian $X_{m,n}$. It is the number

$$degX_{m,n} = (x_1 + \cdots + x_m)^{mn}[X_{m,n}].$$

Schubert [16] determined this number as a special case of the degrees of Schubert varieties [16](26), using an inductive method. Van der Waerden [17]((6) and (7)) reproved Schubert's results, in particular the formula

$$(13) \qquad degX_{m,n} = \frac{(mn)!1!2!\ldots(m-1)!}{n!(n+1)!\ldots(n+m-1)!}$$

A Grassmannian has a Young diagram, for example if $m = 3$, $n = 4$



Each of the mn square boxes has coordinates i, j (with $1 \leq i \leq m$ and $1 \leq j \leq n$) and a hook of length $i + j - 1$.

Formula (13) can be rewritten

$$degX_{m,n} = \frac{(mn)!}{\text{product of all hook lengths}}$$

In our example which is also an example calculated by Schubert [16]

$$degX_{3,4} = \frac{12!}{6!5!12} = \binom{11}{5} = 462$$

Schubert's general formula for the degree of a Schubert variety can also be written in terms of hooks of the Young diagram. Van der Waerden notes ([17] p. 204) that these degrees coincide with the degrees (dimensions) of irreducible representations of the symmetric group S_N where N is the dimension of the Schubert variety, equal to the number of boxes in the Young diagram. My information on hooks and on representations of the symmetric group comes from the book by Sagan [18]. In this book also the story about the hooks is told (Frame, Robinson, Thrall [19]).

As a final remark I mention formula (9) in my paper [20] (compare §6 above and see [10] Part II §24.10). For the case $\frac{U(m+n)}{U(m)\times U(n)}$ this formula says

$$deg\frac{U(m+n)}{U(m)\times U(n)} = \frac{(mn)!}{\prod \mu(b_k)}$$

where the complementary roots b_k are the positive roots of $U(m+n)$ which do not belong to $U(m) \times U(n)$. The b_k correspond to the mn boxes in the diagram in a natural way such that $\mu(b_k)$ becomes the hook length of the box.

The first Chern class of $X_{m,n}$ is $(m+n)(x_1 + x_2 + \cdots + x_m)$. If we substract from $m+n$ a hook length, we get again a hook length. This corresponds to a symmetry of the diagram which one can associate to Serre duality, for the following reason: Let D be a divisor on $X_{m,n}$ with characteristic class $x_1 + x_2 + \cdots + x_m$. Then the hook lengths are the negative roots of the RR-polynomial $\chi(X_{m,n}, rD)$, just as in formula (8) in the case $m = 2$. By the Plücker embedding, the Grassmannian $X_{3,4}$ becomes a smooth 12-dimensional submanifold of degree 462 in $\mathbb{P}_{34}$. We have

$$\chi(X_{3,4}, D) = \prod_{\text{hooks}} \frac{\text{hook length} + 1}{\text{hook length}} = 35 = \binom{7}{3}.$$

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