Introduction to the theory of vector bundles and K-theory

Lectures at the universities of Amsterdam and Bonn

by

Prof.dr. F. Hirzebruch

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notes by M. Hazewinkel (Amsterdam)
D. Erle (Bonn)

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Preface

In the fall of 1964 I lectured for one month at the University of Amsterdam. The purpose was to give an elementary account of the theory of vector bundles and the cohomology theory derived from it. In the lectures I gave the elementary proof of Atiyah and Bott [5] for Bott's periodicity theorem with all details. This theorem is fundamental for the development of the theory. The proof is only partially reproduced in these notes.

It is possible and also enjoyable to develop the theory without using ordinary cohomology theory, keeping it in this way selfcontained and elementary, based only on the notion of vector bundle. In order to carry through this elementary method and to reach interesting applications quickly, one has to use the Adams operation Ψ_k. I had at my disposal private notes of Adams on the splitting of a λ-ring which he worked out during the Bonn Arbeitstagung 1964. I also used an unpublished manuscript of Atiyah on the non-existence of elements of Hopf invariant one.

After my lectures in Amsterdam I gave more or less the same lectures in Bonn. At that time the first portion of Atiyah's lectures on K-theory (Harvard, Fall term 1964) became available, which in some instances were used in the preparation of these notes.

My sincere thanks are due to M. Hazewinkel and D. Erle for writing the notes.

Bonn, June 24, 1965 F. Hirzebruch.

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Contents:

A. Vector bundles

(A.1) Definition

A <u>complex vectorbundle</u> over a topological space X, is a topological space E, with a continuous map $p : E \to X$ (called the projection on X) such that.

1) $p^{-1}(x)$ is a complex vectorspace of finite dimension for all $x \in X$.
2) (local triviality) For all $x \in X$, there is a neighborhood U of x, and a natural number n such that $p^{-1}(U)$ is isomorphic to $U \times \mathbb{C}^n$, that is, there is a homeomorphism $\phi : p^{-1}(U) \to U \times \mathbb{C}^n$ such that the following diagram commutes.

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\quad\phi\quad} & U \times \mathbb{C}^n \\ \Big\downarrow p & & \Big\downarrow pr \\ U & \xrightarrow{\quad id\quad} & U \end{array}$$

while $\phi_x = \phi|p^{-1}x$ is an isomorphism of vectorspaces (The left arrow is $p|p^{-1}(U)$, the right arrow is the projection $U \times C^n \to U$).

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Notation: $p^{-1}(x) = E_x$.

Remarks: 0. p is surjective, because each E_x is a vectorspace and therefore non void

1. condition 2 implies that on a connected component of X the dimension of $p^{-1}(x)$ is constant. This number is called the <u>dimension</u> of the bundle on this component.
2. We will often write for a vectorbundle $E \xrightarrow{p} X$ simply E.

(A.2) Homomorphism, Isomorphisms, Subbundles.

A <u>homomorphism of vectorbundles</u> over X $\psi : E \to F$ is a continuous map such that the following diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{\quad\psi\quad} & F \\ & {\scriptstyle p_E}\searrow \quad \swarrow{\scriptstyle p_F} & \\ & X & \end{array}$$

and such that $\psi|E_x$ is a homomorphism of vectorspaces. We will denote the set of homomorphisms $E \to F$ by $\mathcal{H}om\,(E,F)$.

A homomorphism $\psi : E \to F$ is an isomorphism if ψ_x is bijective for all $x \in X$.

The set of isomorphisms $E \to F$ will be denoted Iso (E, F). The vectorbundles over X, with the homomorphisms of vectorbundles constitute a category, which we will denote by $\mathcal{V}(X)$.

The proof that the above definition of isomorphism checks with the general definition of an isomorphism in a category applied to $\mathcal{V}(X)$, is left to the reader.

(the continuity of the (well-defined) inverse is a local question, so we have only to consider the inverse of a map $X \times \mathbb{C}^n \xrightarrow{1\times A_x} X\times\mathbb{C}^n$, where A_x is a n^2-matrix dependin continuously on $x\in X$, then the inverse map A^{-1} is $X\times\mathbb{C}^n \xrightarrow{1\times A_x^{-1}} X\times\mathbb{C}^n$, and $x \rightsquigarrow A_x^{-1}$ is a continuous map, being the composition of the maps $X \to Gl(n, \mathbb{C}) \xrightarrow{inv} Gl(n, \mathbb{C})$. q.e.d. (inv = taking the inverse)).

A vector bundle E over X is called a <u>subbundle</u> of the vector bundle F over X, if $E \subset F$ and the inclusion is a homomorphism of vector bundles.

(A.3) Sections

The bundle $X \times \mathbb{C}^n$ is called the trivial bundle. If we have a bundle E on X, a map $s : X \to E$ such that ps=id is called a section; it is said to be a nowhere vanishing section if $s(x) \neq 0$ for every $x\in X$.

A bundle E is called of dimension n, if $p^{-1}(x)$ is of dimension n for all $x\in X$. A bundle is called a linebundle, if it has dimension 1. We then have the following lemma.

Lemma: Let E be a bundle of dimension n, such that there are n everywhere, linearly independent sections on X. (hence nowhere vanishing), then the bundle E is isomorphic to the trivial bundle $X \times \mathbb{C}^n$.

Proof: The isomorphism $\phi : X \times \mathbb{C}^n \to E$ is given by $\phi_x = A_x$, where A_x is the matri

$$A_x = \begin{pmatrix} S_1^1(x) & \dots & S_n^1(x) \\ S_1^2(x) & \dots & \\ \vdots & & \\ S_1^n(x) & \dots & S_n^n(x) \end{pmatrix}$$

and $S_k^j(x)$ is the j^{th} component of the vector $S_k(x)$.

In particular we have that a linebundle, which has a nowhere vanishing section is trivial.

(A.4) Lifting of vectorbundles.

Let E be a complexvectorbundle over X, and $f : Y \to X$ a continuous map.

We define the <u>lifted</u> or <u>induced bundle</u> f^*E over Y by $(f^*E) =$

$= \{(y, v) \mid (y,v) \in Y \times E \wedge v\in E_{f(y)}\}$ with the projection map $(y,v) \rightsquigarrow y$; the topology

of f^*E is the topology it inherits as a subspace of $Y \times E$.
Condition 1) of the definition of a vectorbundle (A.1) is clearly satisfied, and it is easily checked that if $U \quad X$ is a neighborhood such that $p^{-1}U$ is isomorphic to $U \times \mathbb{C}^n$, then $f^{-1}(U)$ is a neighborhood on Y above which f^*E is trivial; which takes care of condition 2.

Remark: 1. f^*E is the fibred product (pullback) of the diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

in the category of topological spaces and continuous maps (Top).

2. If S is an arbitrary subspace of X, then $p_E^{-1}(S)$ has a natural structure of vector bundle over S, the so-called restriction of E onto S, notation $E|S$. There is a canonical isomorphism $E|S \simeq i^*E$, where $i : S \to X$ is the inclusion.

3. If $g : Z \to Y$ and $f : Y \to X$ are maps, then

$$(fg)^*E \simeq g^*f^*E.$$

(A.5) (cf [12])

Let $\mathcal{V}$ be <u>the category of complex vectorspaces of finite dimension and linear maps</u>.
A p-times covariant, q-times contravariant functor $\mathfrak{A} : \mathcal{V}^{p+q} \longrightarrow \mathcal{V}$ is called <u>continuous</u> if the map $\bar{\mathfrak{A}}$ induced by $\mathfrak{A}$:

$$\bar{\mathfrak{A}} : Hom(E_1, E_1') \times \ldots \times Hom(E_p, E_p') \times Hom(F_1', F_1) \times \ldots \times Hom(F_q', F_q)$$

$$\longrightarrow Hom(\mathfrak{A}(E_1, \ldots, E_p | F_1, \ldots, F_q), \mathfrak{A}(E_1', \ldots, E_p' | F_1', \ldots, F_q'))$$

is continuous. The functor is called <u>multilinear</u> if this map is multilinear.

<u>Examples</u>: 1. The two times covariant functor $\oplus$, which associates to two vectorspaces X, Y the direct sum $X \oplus Y$ is a continuous, but not bilinear functor. (To a pair matrices $(A, B) \in Hom(X, X') \times Hom(Y, Y')$ is associated the matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in Hom(X \oplus X', Y \oplus Y')$ and $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + \begin{pmatrix} A' & 0 \\ 0 & B \end{pmatrix} \neq \begin{pmatrix} A+A' & 0 \\ 0 & B \end{pmatrix}$.)

2. The 1-time contravariant functor $Hom(-, \mathbb{C})$ which associates to a vector space A, its dual space A^* is linear.

<u>Remark</u>: Multilinearity implies continuity, since we work with vector spaces of <u>finite</u> dimension.

(A.6) Theorem

Given a continuous functor p-times covariant, q-times contravariant

$\mathfrak{a} : \mathcal{V}^{p+q} \to \mathcal{V}$, then there exists one and only one function which attaches to any topological space X a functor $\mathfrak{a}_X : \mathcal{V}(X)^{p+q} \to \mathcal{V}(X)$, p-times covariant q-times contravariant, such that

1) $(\mathfrak{a}_X(E_1,\ldots,E_p|F_1,\ldots,F_q))_x = \mathfrak{a}((E_1)_x,\ldots,(E_p)_x|(F_1)_x,\ldots,(F_q)_x)$.

2) For homomorphisms $e_i : E_i \to E'_i$ $i=1,\ldots,p$; $f_j : F'_j \to F_j$ $j=1,\ldots,q$ we have

$$(\mathfrak{a}_X(e_1,\ldots,e_p)|f_1,\ldots,f_q))_x = \mathfrak{a}((e_1)_x,\ldots,(e_p)_x|(f_1)_x,\ldots,(f_q)_x) .$$

3) If $Y \subset X$ then

$$\mathfrak{a}_Y(E_1|Y,\ldots, E_p|Y \mid F_1|Y,\ldots, F_q|Y) = (\mathfrak{a}_X(E_1,\ldots,E_p \mid F_1,\ldots,F_q))|Y .$$

4) If E_i, $i=1,\ldots,p$; F_j $j=1,\ldots,q$ are trivial bundles, then $\mathfrak{a}_X(E_1,\ldots,E_p| F_1,\ldots,F_q$ is a trivial bundle.

If we have two continuous functors $\mathfrak{a}, \mathfrak{a}' : \mathcal{V}^{p+q} \to \mathcal{V}$ and a natural transformation $\phi : \mathfrak{a} \to \mathfrak{a}'$, then ϕ induces a natural transformation $\phi_X : \mathfrak{a}_X \to \mathfrak{a}'_X$; if ϕ was a natural equivalence, then ϕ_X is a natural equivalence.

Proof: [12]

(A.7) Examples

Using (A 6) we can construct

$E \oplus F$ the direct sum of two vector bundles (Whitney sum).

$E \otimes F$ The tensor-product of two vectorbundels

Hom (E, F) The bundle which has as fiber at x $Hom(E_x, F_x)$.

$E^+ = Hom(E, \mathbb{C})$.

We have for vectorspaces B , A a natural equivalence

$$Hom(A ,B) \xrightarrow{\sim} A^* \otimes B$$

given by the isomorphism $\phi : A^* \otimes B \to Hom(A, B)$

$f \otimes b \to g$ where $g(a) = f(a).b$

(this is an isomorphism, since A is of finite dimension, which is essential.)

So by the (A, 6) we also have a natural equivalence of vector bundles

$$Hom(E, F) \xrightarrow{\sim} E^* \otimes F$$

Remark: If E is vectorbundle over X, we denote by ΓE the set of sections over X.

With this notation we have

$$\Gamma(Hom(E, F)) = \mathcal{H}om(E, F).$$

For an element of $\mathcal{H}om(E,F)$ is a set of homomorphisms $f_x \in Hom(E_x, F_x)$ depending continuously on $x \in X$. This is exactly an element in $\Gamma\mathrm{Hom}(E, F)$ and vice versa.

(A. 8) The exterior powers.

V is a finite dimensional vector space over $\mathbb{C}$. Let $V^n = V \otimes \ldots \otimes V$ (n factors). Then the group S_n of permutations of n-elements acts on V^n in the following manner. If $\sigma \in S_n$, $v = v_1 \otimes \ldots \otimes v_n \in V^n$ then

$$\sigma v = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} \quad .$$

Let Q^n be the subspace of V^n generated by the elements of the form

$$\sigma v - \mathrm{sign}(\sigma) v, \qquad v \in V^n \quad , \quad \sigma \in S_n$$

(sign(σ) is the parity of σ, i.e. sign $\sigma = \pm 1$ depending on whether σ is an even or odd permutation.)

Then we definie

$$\lambda^n V = V^n / Q^n \quad , \qquad \lambda^0 V = \mathbb{C} \quad .$$

The λ^i are clearly covariant functors from the category of $\mathbb{C}$-vectorspaces to the category of $\mathbb{C}$-vectorspaces. They are moreover continuous and there is a functorial isomorphism

(A8.1) $$\lambda^n(V \oplus W) = \sum_{j+i=n} \lambda^i(V) \otimes \lambda^j(W) \ .$$

This means that we have analogous functors λ^i for vector bundles on X by (A.6). If E is a vectorbundle over X then $\lambda^i(E)$ is a vector bundle with fiber $(\lambda^i E)_x = \lambda^i(E_x)$ and the above isomorphism (A8.1) remains true.

(A.9) Topological lemmas

1. (The Tietze extension theorem)
Let X be a normal topological space, $Y \subset X$ a closed subspace. Let $f : Y \to [-1, 1]$ be a continuous function, then there exists a continuous function $g : X \to [-1, 1]$ such that $g|Y = f$.
(That f is a function to $[-1, 1]$ is not essential and may be replaced by for example $f : Y \to \mathbb{R}$).

2. (Partion of unity)
Let X be a normal space, and $\{U_\alpha\}$ a locally finite open covering of X, then there exists real valued functions $\phi_\alpha : X \to \mathbb{R}$ such that

1° $\phi_\alpha(x) \geq 0$ for all α, all $x \in X$

2° $\phi_\alpha(x) = 0$ for $x \in X$ outside some closed subset of U_α.

3° $\sum_\alpha \phi_\alpha(x) = 1$ for all $x \in X$.

For simplicity's sake, we will restrict ourselves in the following to compact Hausdorff spaces (unless otherwise stated) although most of the reasoning can be adapted to the case where X is paracompact.

(A. 10) Lemma

Let $f : E \to F$ be a homomorphism of vectorbundles on X and suppose the function

$$x \rightsquigarrow \text{rank } (f_x : E_x \to F_x)$$

is continuous, then there is for all $x \in X$ a neighborhood U of x and m sections $r_1, \ldots, r_m$ over U, n sections $s_1, \ldots, s_n$ of F over U such that

(i) for all $y \in U$ $r_1(y), \ldots, r_m(y)$ is a basis for E_y

(ii) for all $y \in U$ $s_1(y), \ldots, s_n(y)$ is a basis for F_y

(iii) for all $y \in U$ and all $(v_1, \ldots, v_m) \in \mathbb{C}^m$

$$f(v_1 \cdot r_1(y) + \ldots + v_m \cdot r_m(y)) = v_1 \cdot s_1(y) + \ldots + v_k \cdot s_k(y)$$

where $k = \text{rank } (f_y : E_y \to F_y)$.

Proof: [12] (2.4)

(A.11) Let $f : E \to F$ be a homomorphism of vectorbundles in X such that the function

$$x \rightsquigarrow \text{rank}(f_x : E_x \to F_x)$$

is continuous, then there exists exactly one subbundle E' of E such that ($i : E' \to E$ is the injection)

$$0 \longrightarrow E' \xrightarrow{\ i\ } E \xrightarrow{\ f\ } F$$

is exact. (E' = Ker f).

Proof. Let $x \in X$, U, $r_1, \ldots, r_m$, $s_1, \ldots, s_n$, k be as in (A.10) so for all $y \times (v_1, \ldots, v_m) \in U \times C^n$

(*) $$f(v_1 \cdot r_1(y) + \ldots + v_m \cdot r_m(y)) = v_1 \cdot s_1(y) + \ldots + v_k \cdot s_k(y)$$

Let E'_y be the vectorspace generated by $r_{k+1}(y), \ldots, r_m(y)$.

Let $E' = \bigcup_y E'_y$ with the induced topology as a subspace of E. By (A.3) E' is trivial above U. So E' is a vectorbundle on X. The exactness of the sequence follows immediately out of (*).

Corollary: Let $f : E \to F$ be surjective then Ker f exists and

$$0 \to \text{Ker } f \longrightarrow E \longrightarrow F \longrightarrow 0$$

is exact.

(A. 12) Let E, $F \in \mathcal{V}_X$, X paracompact, hausdorff and let $E \xrightarrow{f} F \longrightarrow 0$ be an exact sequence in $\mathcal{V}_X$.

Then there is a homomorphism of vectorbundles $g : F \to E$ such that fg = id.

Proof: By (A.10) there are for every $x \in X$, a neighborhood U of x and sections $r_1,\dots,r_m$ of E over U, $s_1,\dots,s_n$ of F over U such that there $r_1,\dots,r_m$ are linearly independent over U, as are the $s_1,\dots,s_n$ and such that

$$f(v_1 \cdot r_1(y)+\dots+v_n \cdot r_m(y)) = v_1 \cdot s_1(y)+\dots+v_m \cdot s_n(y)$$

$(m = \dim E_x \geq \dim F_x = n = k)$

For all $y \in U$ we define $k_U : p_F^{-1}(U) \to E$ by

$$k_U(v_1 \cdot s_1(y)+\dots+v_n \cdot s_n(y)) = v_1(r_1(y)+\dots+v_n \cdot r_n(y) .$$

Then for all $v \in p_F{-1}(u)$, $f \circ k_U(v) = v$ and k_U is a homomorphism of $F|U$ in $E|U$.

In this way we find X being paracompact a locally finite covering $\{U_i\}_{i \in I}$ of X such that there is for all $i \in I$ a homomorphism $k_i : F|U_i \to E|U_i$

$$f \circ k_i(v) = v \qquad \text{for all } v \in p_F^{-1}(U_i) .$$

Let $\{\phi_i\}_{i \in I}$ be a partition of unity relative to the covering $\{U_i\}$. Define $g : F \to E$ by

$$g(v) = \sum_{i \in I} \phi_i(x).k_i(v) \qquad x = p_F(v) ,$$

then g is a homomorphism of vectorbundles and fg=id. q.e.d.

Corollary:

Every exact sequence of vectorbundles over X

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

splits, and therefore $E \simeq E' \oplus E''$.

(A. 13) Proposition

Let X be a compact space, E a vectorbundle over X, Y a closed subset of X, and s a section of E over Y, then there is an extension of s to X.

Proof: Let $x \in X$ be any point and U(x) a neighborhood of x such that E is trivial above $\overline{U(x)}$. A section of $E|\overline{U(x)}$ is just a continuous map $\overline{U(x)} \longrightarrow \mathbb{C}^n$ (Check!) $Y \cap \overline{U}$ is closed in $\overline{U}$, and $s|Y \cap \overline{U}$ is a continuous map $Y \cap \overline{U} \longrightarrow \mathbb{C}^n$. $\overline{U}$ is compact, hence normal. Applying A.9.1 we get an extension t to U(x) of $s|Y \cap U$. Finitely many of the neighborhoods U(x) cover X, (compactness), say X is coverd by $\{U_\alpha\}$, with t_α the extension to U_α of . $s|Y \quad U_\alpha$. Let $\{\phi_\alpha\}$ be a partition of unity relative to $\{U_\alpha\}$. We define

$$s_\alpha(x) = \phi_\alpha(x)\, t_\alpha(x) \qquad \text{for } x \in U_\alpha$$
$$s_\alpha(x) = 0 \qquad \text{for } x \notin U_\alpha \quad .$$

Then the s_α are sections of E over X, and it is easy to check that $s = \sum_\alpha s_\alpha$ is an extension of s. q.e.d.

(A.14)
Let X be a compact space, E, $F \in \mathcal{V}_X$, Y is closed subspace of X, and $t \in \mathrm{Hom}(E|Y, F|Y)$ su that t_y is bijective for all $y \in Y$. Then t can be extended as an <u>isomorphism</u> to some neighborhood U of Y.

<u>Proof</u>: $t \in \mathrm{Hom}(E/Y, F/Y) = \Gamma\mathrm{Hom}(E/Y, F/Y) = \Gamma(\mathrm{Hom}(E, F)/Y)$
(A.7) Applying (A. 13) we see that there is an extension of t to a homomorphism $s : E \to F$. Define U to be

$$U = \{x \mid s_x : E_x \to F_x \text{ is bijective}\}.$$

Then $U \supset Y$ if $x \in U$, there is a neighborhood V(x) such that both bundles are trivial above (V(x), $s|p_E^{-1}(V)$ is essentially a map $g : V \longrightarrow \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$.
$U \cap V = \{z \mid z \in V \wedge \det g(z) \neq 0$ is open in V, hence U is open. q.e.d.

(A.15) Let X, Y be compact spaces, $E \in \mathcal{V}_Y$, and let $f, g : X \to Y$ be continuous maps such that f is homotopic to g. The lifted bundles f^*E and g^*E are then isomorphic.

<u>Proof</u>:Let $F : X \times I \to Y$ be a homotopy from f to g and set $F_t(x) = F(x, t)$. Let $\pi: X \times I \to X$ be the canonical projection on X. We compare the vectorbundles $\pi^* F_t^*(E)$ and $F^*(E)$ over $X \times I$. These vectorbundles are isomorphic on $X \times \{t\}$. Applying (A.14) they are isomorphic on an open neighborhood V of $X \times \{t\}$in $X \times I$. Now X is compact so there is a neighborhood U of $t \in I$ such that $X \times U \subset V$. Therefore $F_{t'}^*(E) \simeq F_t^*(E)$ for all $t' \in U$. In view of the fact that I is compact and connected this proves that $f^*E \simeq F_0^*E \simeq F_1^*E \simeq g^*E$. q.e.d.

Let B(X) be the set of isomorphisms classes of vectorbundles over any compact space X. The following is an immediate consequence of the theorem above:

If $f : X \to Y$ is a homotopy equivalence of compact spaces, then the function $f^* : B(Y) \to B(X)$, induced by the lifting of vectorbundles, is bijective.

Corollary 1: If X is contractible, then

$$B(X) \simeq B(\text{point}) \simeq \{\text{non-negative integers}\} \ .$$

Corollary 2: If E is a vectorbundle over $X \times I$ (I the unit interval), $i : X \to X \times$ I the natural injection onto $X\times\{0\}$, and $\pi: X \times I \to X$ the projection, then $\pi^* i^* E \simeq E$.

(A.16) Metrics

Let X be a compact space, $E\in\mathcal{V}_X$. The collection of all bilinear Hermilian forms on a vectorspace from a vector space. Usingthe construction of (A.6) we get a vectorbundle $\mathfrak{h}$ over X, with $\mathfrak{h}_x$ = vectorspace of bilinear Hermitian forms on E_x. A section s in $\mathfrak{h}$ over X such that s(x) is positive definit for all $x\in X$ is called a metric in the complex vectorbundle E.

Proposition:

There exists a metric in every complexvectorbundle E.

Proof: Let $\{U_\alpha\}$ be a covering of X such that $E|U_\alpha$ is trivial for all α.
We can find a metric s_α in $E|U_\alpha$. Let $\{\phi_\alpha\}$be a partition of unity relative to the covering $\{U_\alpha\}$ define

$$s'_\alpha(x) = s_\alpha(x)\,\phi_\alpha(x) \qquad x\in U_\alpha$$
$$= 0 \qquad x\notin U_\alpha \quad .$$

Then s'_α is a section of $\mathfrak{h}$ and if $s'_\alpha(x) \neq 0$ then $s'_\alpha(x)$ is a positive definite Hermitian form . The sum of two positive definite Hermition forms is again positive definite.

Define $s = \sum_\alpha s'_\alpha$

Then s is a metric. q.e.d.

(A.17) Clutching of vectorbundles

Let X be a compact space, X_1, X_2 two compact subsets of X such that $X_1 \cup X_2 = X$. Let $A = X_1 \cap X_2$. Suppose we are given

i) a vectorbundle E_i over X_i $i=1,2$

ii) $\phi \in \mathrm{Iso}(E_1|A, E_2|A) \subset \Gamma\mathrm{Hom}(E_1|A, E_2|A)$,

Then we construct a new bundle $E = E_1 \cup_\phi E_2$ (clutching E_1 and E_2 by ϕ) as follows:

$$E_x = E_{1x} \qquad \text{if } x\in E_1$$
$$E_x = E_{2x} \qquad \text{if } x\in E_2$$

where for $x\in A$ E_{1x} and E_{2x} are identified by means of ϕ (v "=" $\phi(v)$).

E has the topology obtained by considering it as a quotient space of $E_1 \dot\cup E_2$ (the disjoint union of E_1 and E_2). Using (A.14) we check the local triviality for points $x\in A$. (For points $x\notin A$ it is immediately clear, A being closed).

We remark that

1. If E is a bundle over X, then $E|X_1 \underset{id_A}{\cup} E|X_2 = E$.

2. If E_1, E_1' are bundles over X_1, E_2, E_2' bundles over X_2,
$\phi \in Iso(E_1|A, E_2|A)$, $\phi' \in Iso(E_1'|A, E_2'|A)$ then there is a natural isomorphism
$(E_1 \underset{\phi}{\cup} E_2) \oplus (E_1' \underset{\phi'}{\cup} E_2') \simeq (E_1 \oplus E_1') \underset{\phi\oplus\phi'}{\cup} (E_2 \oplus E_2')$

3. The same remark holds for the tensorproduct.

4. $(E_1 \underset{\phi}{\cup} E_2)^* \simeq E_1^* \underset{(\phi^*)^{-1}}{\cup} E_2^*$

This follows from the following general property of the clutching operation with respect to continuous functors).

(A.17) continued

Let $\mathfrak{A}$ be a continuous functor, say in two variables, covariant in the first, contravariant in the second one, $\mathfrak{A} : \mathcal{V}^2 \longrightarrow \mathcal{V}$. (The general case is a complete analogue.) Given E_i, F_i vector bundles over X_i $(i=1,2)$, where X, X_1, X_2 compact, $X=X_1 \cup X_2$ $X_1 \cap X_2 = A$ as above, and isomorphisms $\phi \in Iso(E_1|A, E_2|A)$, $\psi \in Iso(F_1|A, F_2|A)$.

<u>Proposition:</u>
There is a canonical isomorphism of vectorbundles

$$\mathfrak{A}_{X_1}(E_1, F_1) \underset{\mathfrak{A}_A(\phi,\psi^{-1})}{\cup} \mathfrak{A}_{X_2}(E_2, F_2) \simeq \mathfrak{A}_X(E_1 \underset{\phi}{\cup} E_2, F_1 \underset{\psi}{\cup} F_2).$$

Proof: $\mathfrak{A}_A(\phi,\psi^{-1})$ is a clutching function because $\mathfrak{A}_A$ is a functor.

A bijection of the total spaces of the two vector bundles which is linear on each fibre, is obtained in this way:

$$\mathfrak{A}_X(E_1 \underset{\phi}{\cup} E_2, F_1 \underset{\psi}{\cup} F_2) = \mathfrak{A}_{X_1}(E_1 \underset{\phi}{\cup} E_2|X_1, F_1 \underset{\psi}{\cup} F_2|X_1) \cup \mathfrak{A}_{X_2}(E_1 \underset{\phi}{\cup} E_2|X_2, F_1 \underset{\psi}{\cup} F_2|X_2),$$

$\mathfrak{A}_{X_i}(E_i, F_i) \simeq \mathfrak{A}_{X_i}(E_1 \underset{\phi}{\cup} E_2|X_i, F_1 \underset{\psi}{\cup} F_2|X_i)$, and two elements $\alpha_i \in \mathfrak{A}_{X_i}(E_i, F_i)$, $i=1,2$, have the same image in $\mathfrak{A}_X(E_1 \underset{\phi}{\cup} E_2, F_1 \underset{\psi}{\cup} F_2)$ under the latter isomorphisms if and only if $\alpha_2 = \mathfrak{A}_A(\phi,\psi^{-1})\ (\alpha_1)$. (Use (A.6).)

As $\mathfrak{A}_X(E_1 \underset{\phi}{\cup} E_2, F_1 \underset{\psi}{\cup} F_2)$ was represented as the union of closed subsets, the so constructed bijection is bicontinuous. q.e.d.

(A.18) Homotopic clutching functions

Let X, X_1, X_2, E_1, E_2, A be as in (A. 17). Two clutching functions $\phi_0,\phi_1 \in \mathrm{Iso}(E_1|A, E_2|A)$ are called homotopic if the maps $\phi_0,\phi_1 : A \to \mathrm{Hom}(E_1|A, E_2|A$ are homotopic by a homotopy F such that F_t is clutching for all t (i.e $F_t \in \mathrm{Iso}(E_1|A, E_2|A) \subset \Gamma\mathrm{Hom}(E_1|A, E_2|A)$ for all t).

For mutated in an other way:

ϕ_0,ϕ_1 are called homotopic, if there is a continuous set of clutching functions F_t, $t \in [0,1]$ connecting them.

Proposition:

If ϕ_0,ϕ_1 are homotopic clutching functions, then $E_1 \underset{\phi_0}{\cup} E_2 \simeq E_1 \underset{\phi_1}{\cup} E_2$

Proof: Lift E_1 to $X_1 \times I$, E_2 to $X_2 \times I$, let F be a homotopy between ϕ_0,ϕ_1. We can use F to clutch E_1' and E_2' over $A \times I$ ($E_i' = \pi_i^* E_i$, π_i is the projection $X_i \times I \to X_i$). Then $E_1 \underset{\phi_0}{\cup} E_2 \simeq E_1' \underset{F}{\cup} E_2' | X\times\{0\}$ and $E_1 \underset{\phi_1}{\cup} E_2 \simeq E_1' \underset{F}{\cup} E_2' | X\times\{1\}$.

But the maps $X \to X \times I$ defined by $x \rightsquigarrow (x,0)$ and $X \to X \times I$ defined by $x \rightsquigarrow (x,$ are homotopic. Applying (A.15) gives the de sired result.

Remark: Another way to define homotopy of clutching functions is this

$F : (E_1|A) \times I \to E_2|A$ is a homotopy of clutching functions ϕ_0, $\phi_1 \in \mathrm{Iso}\,(E_1|A, E_2|A)$, if $F_t : E_1|A \to E_2|A$ is a clutching function for every $t \in I$ and $F_0 = \phi_0$, $F_1 = \phi_1$.

(This definition yields the same equivalence relation as the one given above).

To prove that $E_1 \underset{\phi_1}{\cup} E_2 \simeq E_1 \underset{\phi_1}{\cup} E_2$ if ϕ_0 and ϕ_1 are homotopic clutching functions, observe that $E \times Y$ is a vectorbundle over $X \times Y$ for any vectorbundle E over X. Now define $G : (E_1|A)\times I \longrightarrow (E_2|A) \times I$ by $G(e,t) = (F(e,t),t)$. It is easy to prove that G is a vectorbundle isomorphism. So we can consider the vectorbundle $E_1 \times I \underset{G}{\cup} E_2 \times I$. The maps f_0, $f_1 : X \to X \times I$ defined by $f_i(x) = (x,i)$ are homotopic. But $f_i^*(E_1 \times I \underset{G}{\cup} E_2 \times I) \simeq E_1 \times I \underset{G}{\cup} E_2 \times I|\ X\times\{i\} \simeq E_1 \underset{\phi_i}{\cup} E_2$, and the proposition follows from (A.15).

(A.19) Collapsing of vectorbundles

Let $Y \subset X$, both compact space, E a vectorbundle over X such that $E|Y$ is trivial. If α: $E|Y \to Y \times \mathbb{C}^n$ is a trivialisation, we define a vectorbundle $E_{(\alpha)}$ over the (compact) space X/Y, by the following collapsing operation: Let $\pi : Y \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be the projection. We introduce an equivalence relation for the points of E:

$$e \sim e' \iff \pi\alpha(e) = \pi\alpha(e') \qquad (e, e' \; E|Y)$$
$$e \sim e' \iff e = e' \qquad (e, e' \; E|X-Y)$$

$E_{(\alpha)}$ is defined to be the quotient space $E/\sim$. There is a natural continuous projection $E_{(\alpha)} \to X/Y$, with respect to which $E_{(\alpha)}$ is a vectorbundle. To prove this we have only to verify the local triviality at the point Y/Y. But $\alpha^{-1} : Y \times C^n \to E/Y$ can be extended to an open neighborhood U of Y as an isomorphism of vectorbundles (apply (A.14)), and U/Y is a neighborhood of Y/Y.

(A.20) Homotopic trivialisations

Let X, Y, E be as in (A.19), $\alpha_0, \alpha_1 : E|Y \to Y \times \mathbb{C}^n$ two trivialisations. A homotopy between α_0 and α_1 is a continuous map $F : (E|Y) \times I \to Y \times \mathbb{C}^n$ such that F_t is a trivialisation for each $t \in I$ and $F_0 = \alpha_0$, $F_1 = \alpha_1$.

<u>Proposition:</u>

If α_0 and α_1 are homotopic trivialisations, then

$$E_{(\alpha_0)} \simeq E_{(\alpha_1)}$$

Proof: A homotopy F between α_0 and α_1 induces a trivialisation $\beta : (E|Y) \times I \to Y \times \mathbb{C}^n \times I$ by $\beta(e,t) = (F(e,t), t)$. The natural projection $\pi : (X/Y) \times I \to X \times I/Y \times I$ is continuous and the injections

$f_0, f_1 : (X/Y) \to (X/Y) \times I$, defined by $f_i(x) = (x,i)$, are homotopic. Applying (A.15), we have

$$E_{(\alpha_0)} \simeq (\pi f_0)^*(E \times I_{(\beta)}) \simeq (\pi f_1)^*(E \times I_{(\beta)}) \simeq E_{(\alpha_1)} .$$

<u>Remark:</u> There is another definition of homotopy of trivalisations which is easily seen to be equivalent to the one given above. The composed map $\alpha_1\alpha_0^{-1} : Y \times \mathbb{C}^n \to Y \times \mathbb{C}^n$ induces a map $g : Y \to Gl(n, \mathbb{C})$. Now α_0 and α_1 are said to be homotopic, if the induced map g is homotopic to a constant map. (Realise that $GL(n,\mathbb{C})$ is pathwise connected.) As an immediate consequence we obtain.

<u>Lemma:</u> If Y is contractible, then any two trivialisations of E|Y are homotopic.

(A.21)

<u>Theorem:</u>

Let X be a compact space, Y a closed subspace of X. The projection $\pi : X \to X/Y$ induces a mapping

$$\pi^* : B(X/Y) \to B(X).$$

(B(X), B(X/Y) is the set of isomorphism classes of vectorbundles over X, X/Y respectively.)

Now assume that Y is contractible. Then π^* is bijective.

Proof: We shall construct a function

$$\phi : B(X) \longrightarrow B(X/Y)$$

such that $\phi \circ \pi^* = id$ and $\pi^* \circ \phi = id$.

If E is a vectorbundle over X, $E|Y$ is trivial by (A.15) Cor. 1. Choose an arbitrary trivialisation α and define $\phi([E]) = [E_{(\alpha)}]$, where the brackets [ ] mean the isomorphism class. By the preceding lemma and proposition ϕ is well-defined. Let F be a vectorbundle over X/Y.

An isomorphism $i : F_{Y/Y} \to \mathbb{C}^n$ defines a trivialisation $\beta = id \times i$ of $\pi^*F|Y = Y \times F_{Y/Y}$. We have a commutative diagramm

$$\begin{array}{ccc} \pi^*F & \subset & X \times F \\ \downarrow & & \searrow \\ (\pi^*F)_{(\beta)} & \subset & X/Y \times F \end{array}$$

where the arrows denote natural projections. As

$$(\pi^*F)_{(\beta)} = \{(x,v) | (x,v) \in X/Y \times F \wedge p_F(v) = x\} \simeq F,$$

$\phi \circ \pi^* = id$ is proved. If E is a vectorbundle over X and α a trivialisation of $E|Y$, define $f : E \to X \times E_{(\alpha)}$ by $f(v) = (p(v), \rho(v))$. ($\rho : E \to E_{(\alpha)}$ is the projection map). Clearly f is an isomorphism of E onto $\pi^*(E_{(\alpha)}) \subset X \times E_{(\alpha)}$.

Thus $\pi^* \circ \phi = id$.

(A. 22)

Let X be a compact space and I the unit interval.

The suspension of X is the following compact space:

$$\Sigma(X) = (X \times I / X \times \{0\}) | X \times \{1\}$$

If $C^-(X) = X \times [0, \frac{1}{2}] / X \times \{0\}$, $C^+(X) = X \times [\frac{1}{2}, 1] / X \times \{1\}$, then $C^-(X) \cup C^+(X) = \Sigma(X)$, $C^-(X) \cap C^+(X) = X \times \{\frac{1}{2}\}$ which we identify with X. $C^-(X)$ and $C^+(X)$ are contractible. $B_n(X)$ will denote the set of isomorphism classes of n-dimensional vectorbundles over X. $[X,Y]$ is the set of homotopy classes of maps $X \to Y$.

Theorem:

There is a natural 1-1-correspondence

$$B_n(\Sigma(X)) \simeq [X, GL(n, \mathbb{C})].$$

Proof: A vectorbundle E over $\Sigma(X)$ has trivial restrictions to $C^-(X)$ and $C^+(X)$. If $\alpha_\pm : E|C^\pm \to C^\pm \times \mathbb{C}^n$ are trivialisations, $(\alpha_-|X) \circ (\alpha_+|X)^{-1} : X \times \mathbb{C}^n \to X \times \mathbb{C}^n$ induces a map $f : X \to GL(n, C)$. As α_+ and α_- are determined up to homotopy of trivialisations by E, f is determined up to homotopy by E. We put $\phi([E]) = [f]$.

On the other hand a map $g : X \to GL(n, \mathbb{C})$ defines a clutching function $\gamma : X \times \mathbb{C}^n \to X \times \mathbb{C}^n$, which can be used to clutch the trivial bundles $C^+ \times \mathbb{C}^n$ and $C^- \times \mathbb{C}^n$ together. Changing g homotopically does not change the isomorphism class of $C^+ \times \mathbb{C}^n \cup_\gamma C^- \times \mathbb{C}^n$. Define $\psi([g]) = |C^+ \times \mathbb{C}^n \cup_\gamma C^- \times \mathbb{C}^n|$.

To verify $\psi \circ \phi = id$ and $\phi \circ \psi = id$, is not difficult. So ϕ and ψ are bijections. q.e.d.

(A.23) Lemma on bundles.

Let X be a compact space, E a complex vectorbundle over X, then there exists a natural number n and a complex vectorbundle F such that $E \oplus F \simeq X \times \mathbb{C}^n$.

Proof: Cover X with a finite number of open sets U_i, above which E is trivial. For all U_i select sections $s_1^i, \ldots, s_{n(i)}^i$ such that $s_1^i(x), \ldots, s_{n(i)}^i(x)$ form a basis for E_x for all $x \in U_i$.

Let $\{\phi_i\}_i$ be a partition of unity relative to $\{U_i\}_i$. We define

$$\bar{s}_j^i(x) = 0 \qquad x \notin U_i$$
$$= \phi_i(x)\, s_j^i(x) \qquad x \in U_i \quad .$$

Then for every $x \in X$, the $\bar{s}_j^i(x)$ form a set of generators for the fibre E_x. Let N be the number of $\bar{s}_j^i$. We define a homomorphism

$$\sigma : X \times \mathbb{C}^N \longrightarrow E$$

by setting $\sigma(x, z_j^i) = \Sigma z_j^i\, \bar{s}_j^i(x) \in E_x$.

This homomorphism is surjective being surjective in every fibre. By (A.11) the kernel F of σ exists and by (A.12) is $E \oplus F \simeq X \times C^N$. q.e.d.

Remark: If X is paracompact of finite dimension, then the lemma is still true (cf [12]).

(A. 24)

We shall establish a classification theorem for $B_n(X)$, X an arbitrary compact space. First some prelinearies:

Consider the set $G_{m,n}$ of n-dimensional vector subspaces of $\mathbb{C}_{m+n}$. We can embed $G_{m,n}$ into End $(\mathbb{C}_{m+n})$ by assigning to each $g \in G_{m,n}$ the uniquely determined orthogonal projection operator (relative to the canonical Hermitian metric of $\mathbb{C}_{m+n}$) which has g as its range. We give $G_{m,n}$ the relative topology. Thus $G_{m,n}$ is compact Hausdorff. Let (n, m+n; n) be the complex (n, m+n)-matrices of rank n. The rows of an element of M(n, m+n; n) span an element of $G_{m,n}$. We thus get a projection $p : M(n, m+n\,; n) \to G_{m,n}$. It is elementary to show that $G_{m,n}$ has the identification

topology relative to p. The spaces $G_{m,n}$ are called Grassmann manifolds.
$G_{m,n}$ is indeed a manifold, but we neither need nor prove this. We refer to |13| where real Grassmann manifolds are treated.

We now define the universal bundle $E_{m,n}$ over $G_{m,n}$.
The total space is $E_{m,n} = \{(g,c) \mid g \in G_{m,n} \wedge c \in \mathbb{C}_{m+n} \wedge c \in g\}$. The projection of this bundle is to be the projection onto the first factor. Local triviality is proved as follows:

Consider $G_{m,n}$ as a subspace of $\mathrm{End}(\mathbb{C}_{m+n})$. Let $x \in G_{m,n}$. Choose a homomorphism $K : \mathbb{C}_n \to \mathbb{C}_{m+n}$ such that $x \circ K$ has rank n. Then there is a neighborhood U of x such that $u \circ K$ has rank n for all $u \in U$. The map $(u, c) \to (u, (u \circ K)(c))$ defines an isomorphism of $U \times \mathbb{C}_n$ onto $E_{m,n}|U$.

Now Let us leave n fixed. The injection $j_m : C_{m+n} \to C_{m+n+1}$, defined by $j_m(z_1,\dots,z_{m+n}) = (z_1,\dots,z_{m+n}, 0)$, induces a continuous map $i_m : G_{m,n} \to G_{m+1,n}$

Lemma: $i_m^* E_{m+1,n} \simeq E_{m,n}$

Proof: Remember that by definition $E_{m,n} \subset G_{m,n} \times \mathbb{C}_{m+n}$ and

$i_m^* E_{m+1,n} \subset G_{m,n} \times G_{m+1,n} \times \mathbb{C}_{m+n+1}$. $f : E_{m,n} \to i_m^* E_{m+1,n}$, defined by

$f(g,c) = (g, i_m g, j_m c)$, is the required isomorphisme.

(A.25)

Let X be an arbitrary compact space. $[X, G_{m,n}]$ denotes the set of homotopy classes maps from X to $G_{m,n}$. For a fixed n, the sets $[X, G_{m,n}]$ are a direct system, the maps being induced by the i_m's.

Theorem: There is a natural isomorphism

$$\varinjlim_{m\to\infty} [X, G_{m,n}] \simeq B_n(X)$$

Proof:

A map $\alpha : X \to G_{m,n}$ induces a vectorbundle $\alpha^* E_{m,n}$ over X. In view of A.15 this defines a function $[X, G_{m,n}] \to B_n(X)$. If the homotopy classes of $\alpha : X \to G_{m,n}$ and $\beta: X \to G_{m+1,n}$ represent the same element of $\varinjlim_{m\to\infty}[X, G_{m,n}]$, then β and $i_m \circ \alpha$ are homotopic, hence $\beta^* E_{m+1,n} \simeq \alpha^* i_m^* E_{m+1,n} \simeq {}^* E_{m,n}$ by the preceding lemma.

So we have defined a function $\Psi: \varinjlim_{m\to\infty} [X, G_{m,n}] \to B_n(X)$. Now we define a function $\phi: B_n(X) \to \varinjlim_{m\to\infty} [X, G_{m,n}]$ which will turn out to be the inverse of Ψ. Let E be an n-dimensional vectorbundle over X. There is a natural number m and a surjective homomorphism of vectorbundle $f : X \times \mathbb{C}_{m+n} \to E$ by A.23. Putting $\bar{f}(x) = (\operatorname{Ker} f_x)^{\perp}$, the orthogonal complement taken with respect to the canonical Hermitian metric on $\mathbb{C}_{m+n}$, we get a function $\bar{f} : X \to G_{m,n}$. $\bar{f}$ is continuous. (Use A.10 and orthogonalize the sections) $\phi([E])$ is to be the element of $\lim_{\substack{\to \\ m\to\infty}} [X, G_{m,n}]$ represented by the homotopy class of $\bar{f}.\phi$ is well-defined : Given two surjectieve homonorphisms $f_i : X \times \mathbb{C}_{m_i+n} \to E$ $(i=0,1)$, it is sufficient to show that $k_0 \circ \bar{f}_0$ and $K_1 \circ \bar{f}_1$ are homotopic, where the maps $K_i : G_{m_i,n} \longrightarrow G_{m_0+m_1+n,m}$ are compositions of the i_m's. $F_t : X \times \mathbb{C}_{m_0+n} \times \mathbb{C}_{m_1+n} \to E$, $F_t(x, v, w) =$

$(1-t)\ f_0\ (x,v) + t\ f_1(x, w)$, defines a homotopy between $k_0 \circ \bar{f}_0$ and a map $\bar{F}_1 : X \to G_{m_0+m_1+n,n}$ (Note that F_t is surjective for all $t\in I$.) But $\bar{F}_1$ is homotopic to $K_1 \circ \bar{f}_1$, because $K_1 \circ \bar{f}_1 = T \circ \bar{F}_1$, where $T : G_{m_0+m_1+n,m} \to G_{m_0+m_1+n,m}$

is induced by a suitable permutation of the coordinates of

$C_{m_0+m_1+n}$ and therefore homotopic to the identiy.

We have to check $\phi\circ\Psi = id$ and $\Psi\circ\phi = id$.

Let α: $X \to G_{m,n}$ be continuous. $\Psi(|\alpha|) = [\alpha^* E_{m,n}]$. $\alpha^* E_{m,n} \subset X \times G_{m,n} \times \mathbb{C}_{m+n}$.

Define $f : X \times \mathbb{C}_{m+n} \to \alpha^* E_{m,n}$ by $f(x,c) = (x,\alpha\ (x),\ (\ \alpha(x))(c))$ where $\alpha(x)$ is again considered as an orthogonal projection. f is continuous and surjective. $(\operatorname{Ker} f_x)^{\perp} = \alpha(x)$, hence $\bar{f} = \alpha$, which proves $\phi\circ\Psi = id$.

Let $f : X \times \mathbb{C}_{m+n} \to E$ be a homomorphism onto an n-dimensional vectorbundle E over X. $\phi([E]) = [\bar{f}]$. $\Psi\circ\phi\ ([E]) = [\bar{f}^* E_{m,n}]$. For $(x, \bar{f}(x), c) \in \bar{f}^* E_{m,n} \subset X \times G_{m,n} \times \mathbb{C}_{m+n}$ define $j(x, \bar{f}(x), c) = f(x,c)$. Clearly $j : \bar{f}^* E_{m,n} \to F$ is an isomorphism, and $\Psi\circ\phi = id$ is proved. q.e.d.

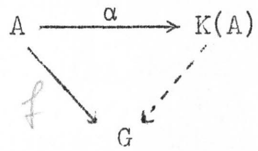
B. Definition and elementary properties of K(X).

(B.1) Definition of the Grothendieck group K(A).

Let A be a commutative monoid. (i.e. A is set with an associative and commutative composition given, which we will call addition and denote by +).
Let F(A) be the free abelian group on A, and R(A) the subgroup generated by the set $\{(a+b)-a-b \mid a,b\ A\}$. Then $K(A) := F(A)/R(A)$ is an abelian group, and we a canonical additive morphism $\alpha: A \longrightarrow K(A)$ obtained by composing the maps $A \rightsquigarrow F(A) \longrightarrow \frac{F(A)}{R(A)} = K(A)$.

K(A) together with this map α have the following universal property:

Every additive map $A \longrightarrow G$ into an abelian group G factorizes uniquely through $\alpha : A \to K(A)$. i.e. every diagram



can be uniquely filled in with a homomorphism $K(A) \longrightarrow G$ to become a commutative diagram.

Proof. Let $f : A \to G$ be an arbitrary additive map. Let $\alpha: A \to K(A)$ be the canonical map. Every element in K(A) is represented by a formal finite sum $\sum_i^{<\infty} n_i a_i$, $n_i \in \mathbb{Z}$, $a_i \in A$. We define $\tilde{f} : K(A) \to G$ by

$$\tilde{f}\left(\sum_i^{<\infty} n_i a_i\right) = \sum_i n_i f(a_i) \in G .$$

If $\sum_i n_i a_i$ and $\sum_i n_i' a_i'$ differ only by a finite sum of elements in R(A), then f being additive $f(\sum_i n_i' a_i') = f(\sum_i n_i a_i)$. By definition of $\tilde{f}$, $\tilde{f}\alpha = f$.
Suppose now, there are two factorizations $\tilde{f}$, $\tilde{f}'$. By definition of K(A), $\alpha(A)$ is a set of generators for K(A). So if $\tilde{f}$, $\tilde{f}'$ are different, there must be a non zero element $\alpha(a) \in \alpha(A) \subset K(A)$ such that $\tilde{f}\,\alpha(a) \neq \tilde{f}'\alpha(a)$ but this is clearly impossible since $\tilde{f}\alpha(a) = f(a) = \tilde{f}'\alpha(a)$.

q.e.d.

If in A there is also defined a multiplication, which is associative, commutative and distributive with respect to addition, (i.e. if A is a commutative semiring) then K(A) becomes a commutative ring. If there is a unity for the multiplication in A, then K(A) becomes a commutative ring with unity.

(B.2) Let A, A' be two monoids, $f : A \to A'$ an additive map. Then $\alpha' f : A \to A' \to K(A')$ is an additive map into an abelian group $K(A')$ and therefore factorizes through $\alpha : A \to K(A)$ to yield the following diagram.

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \alpha \downarrow & & \downarrow \alpha' \\ K(A) & \xrightarrow{K(f)} & K(A') \end{array}$$

So we have associated to $f : A \to A'$ a uniqly determined map $K(f) : K(A) \to K(A')$. It is easily checked that $K(1_A) = 1_{K(A)}$, $K(fg) = K(f)\,K(g)$.
If f is a homomorphism of semirings, then K(f) is a homomorphism of rings.

We can now say that we have constructed a covariant functor K from the category of monoids and additive maps to the category of abelian groups, which when restricted to the category of semirings and homomorphisms of semirings, has its values in the category of rings.

(B.3) Definition of K(X)

Let B(X) be the semiring of isomorphism classes of complex vectorbundles on X.

(X a compact topological space) (Addition "is" the direct sum of vector bundles, multiplication the tensorproduct, the zero element is the zero bundle, the trivial bundle of dimention 1 is the unity for the multiplication.)

Then by (B.1), (B.2) we can consider the ring K(B(X)) which we will write as K(X).

If $f : X \to Y$ is a continuous map. Then f induces a map $f^* : B(Y) \to B(X)$ defined by $E \rightsquigarrow f^*E$ f^*E being the indual bundle over X (A.4).
Moreover $f^*(E \oplus F) \simeq f^*(E) \oplus f^*(F)$ and $f^*(E \otimes F) \simeq f^*(E) \otimes f^*(F)$, so f^* is a homomorphism of semirings. This induces by (B.2) a homomorphism of rings

$$K(f) : K(Y) \longrightarrow K(X)$$

and we have defined a contravariant functor K : <u>comp</u> $\to$ (ring) (<u>comp</u> is the category of compact spaces).
We have already proved (A.15) that $f^*E \simeq g^*E$ if f is homotopic to g. Therefore $K(f) = K(g)$ if f is homotopic to g. In the language of categories this means that the functor K factorizes through the category $\underline{Htp}_{cp}$ (the category of compact spaces and homotopy classes of maps).

(B.4) Line Bundles in K(X).

A vectorbundle E on X is called a line bundle if the dimension of E_x is 1 for all $x \in X$. The tensorproduct of two line bundles is again a line bundle. Moreover if L is a line bundle, then

$$L^{*} \otimes L = \mathrm{Hom}(L, L) \qquad \text{(cf. (A.7))}$$

and there is a nowhere vanishing section in Hom(L,L) in fact the section 1_L, so by (A.3) Hom(L, L) $\simeq$ 1. The trivial bundle of dimension 1 is the unity for the tensorproduct. We have shown that the isomorphism classes of line bundles on X form an abelian group $\mathcal{G}(X) \subset B(X)$.

This group is of course mapped into K(X) and its image there is a subgroup of the group of units of K(X).

(B.5) Lemma

Every element of K(X) can be written in the form $[E] - [n.1]$, where E is a vector bundle on X. $[E]$ the corresponding element in K(X).)

<u>Proof.</u> Every element ξ of K(X) can by definition be written as a finite sum

$$\xi = \sum_i^{<\infty} n_i [E_i], \text{ where the } E_i \text{ are vector bundles on X, } n_i \in \mathbb{Z}.$$

By (A. 23) for every E_i which has a negative coëfficient, we can find a bundle F_i such that

$$-[E_i] = [F_i] - [m_i . 1] \quad .$$

So we can write ξ in the form

$$\xi = \sum_i^{<\infty} n_i([F_i] - [m_i.1]) \text{ with } n_i > 0 \text{ all i.}$$

Taking the direct sum of all the F_i's we get the desired representation of ξ.

q.e.d

(B.6) Stable equivalence

Two vectorbundles E,F on X are called <u>stably equivalent</u>, if there exist natural numbers n,m such that $E \oplus n.1 \simeq F \oplus m.1$.

This is an equivalence relation in the set of isomorphism classes of vector bundles as is easily checked. Let I(X) denote the set of equivalence classes so obtained.

The direct sum of vector bundles induces an addition in I(X). There is a zero element, represented by 0, (which is stably equivalent to every trivial bundle n.1), and (A.23) shows that there is a negative for every element in I(X). Therefore I(X) is an abelian group.
If $f : X \to Y$ is continuous, $K(f) : K(Y) \to K(X)$ is defined. Define $I(f) : I(Y) \to I(X)$ in the obvious way. Then I is a contravariant functor from the category of compact spaces to the category of abelian groups.

(B.7) Lemma

Let X be compact and connected. Let E, F be vectorbundles on X, then [E] = [F] in K(X) if and only if there exists a natural number n such that

$$E \oplus n.1 \simeq F \oplus n.1$$

<u>Proof:</u> If $E \oplus n.1 \simeq F \oplus n.1$ for some n then [E] + [n.1]= [F] + [n.1] and K(X) being a group [F] = [E] follows. Now suppose [E] = [F] . There is a map $f : B(X) \to I(X)$ defined by f(E) = stable equivalence class of E. This map is additive I(X) is a group so f factorizes through K(X)

$$\begin{array}{ccc} B(X) & \xrightarrow{\alpha} & K(X) \\ & \searrow^{f} \quad \swarrow^{\tilde f} & \\ & I(X) & \end{array}$$

This means that if [F] = [E] then E is stably equivalent to F. Now let x be an arbitrary point of X. There is a map $B(X) \xrightarrow{\varepsilon} \mathbb{Z}$ defined by $\varepsilon(E) = \dim E_x$.

This map is additieve and therefore factorizes through K(X) and so if [E] = [F] then $\dim E_x = \dim F_x$.

[E] = [F] implied E, F stably equivalent, so there are natural numbers m,n such that $E \oplus n.1 \simeq F \oplus m.1$ so $\dim(E_x) + n = \dim F_x + m$ but $\dim E_x = \dim F_x$, so m=n. q.e.d.

(B.8)

The foregoing shows that for each $x \in X$ there is an isomorphism $K(X) \xrightarrow{g} I(X) \oplus \mathbb{Z}$ defined by $g([E]) = (\{E\}, \dim(E_x))$, where {E} denotes the stable equivalence class of E.

Let $*$ be the space consisting of one point only, choose a basepoint $x_0 \in X$.

Then we have maps

$$* \underset{p}{\overset{i}{\rightleftarrows}} X$$

$(i(*) = x_0)$ such that pi = id.

So we get the exact sequence

$$K(*) \xrightarrow{p^*} K(X) \longrightarrow K'(X) \longrightarrow 0$$

K'(X) is defined as the cokernel of p^* . We have a map $i^* K(X) \to K(*)$ such that $i^* p^* = id$, so p^* is injective and the sequence splits, $K(*) = \mathbb{Z}$, so

$$K(X) \simeq K'(X) \oplus \mathbb{Z}$$

K'(X) and I(X) are isomorphic by the homomorphism obtained by composing the injection $K'(X) \longrightarrow K(X)$ with the map $K(X) \xrightarrow{f} I(X)$ constructed in (B.7).

If we work in the category of spaces with basepoint we define

$$\tilde{K}(X) = K(X, x_0) = \mathrm{Ker}(i^* : K(X) \longrightarrow K(\{x_0\})) \quad .$$

The foregoing shows that $\tilde{K}(X) \simeq I(X)$: not canonically however as the isomorphism depends on the choice of basepoint.

C. Some remarks on the proof of Bott's periodicity theorem

This chapter contains the formulation and part of the proof of Bott's periodicity theorem which is lateron needed to define a cohomology theory based on the functor K of chapter B. For a complete proof we refer to [5] . We have included some spectral analysis which is required for the proof of the periodicity theorem. We treated the case of Banach spaces though in the application only finite-dimensional vector spaces occur. What we assume to be known about Banach spaces can be found in "J. Dieudonné: Foundations of Modern Analysis (New York and London 1960)".

Some spectral analysis:

(C.1) Let $pr : \mathbb{C}_2 - \{0\} \to P_1\mathbb{C}$ be the natural projection. We consider $\mathbb{C}$ as a subspace of $P_1\mathbb{C}$ by assigning $pr(z,1)$ to $z \in \mathbb{C}$.

Definition: If E, F are complex Banach spaces, $A, B : E \to F$ bounded linear operators, define $\mathrm{Spec}(A, B) \subset P_1\mathbb{C}$ by

$$pr(z, w) \in \mathrm{Spec}(A, B) \iff Az + Bw \quad \text{is nicht invertible with bounded inverse.}$$

(Realise that this makes sense).

Any point of $P_1\mathbb{C}$ which is not in Spec(A, B) is called a regular point (of the pair (A, B)).

Remark: If E, F are finite-dimensional vector spaces, every linear map $E \longrightarrow F$ is a bounded operator. Therefore $pr(z,w) \in \mathrm{Spec}(A, B)$ if and only if the matrix $Az + Bw$ is singular.

(C.2) Lemma:

Spec(A, B) is a compact subset of $P_1\mathbb{C}$.

Proof: It is sufficient to prove that the set of regular points of (A,B) is open in $P_1\mathbb{C}$. Let pr(z, w) be a regular point of (A, B). Without loss of generality we may assume $w \neq 0$ and even $w=1$.

Let C_z denote the real number $C_z = \|(Az + B)^{-1}\|$.

Let $z' \in \mathbb{C}$ be such that $|z-z'| < \frac{1}{C_z \|A\|}$. We prove that z' is a regular point of (A, B). The series

$$S = \sum_{k=0}^{\infty} (Az + B)^{-1} (z-z')^k \{A(Az + B)^{-1}\}^k$$

is convergent if $|z-z'| < \frac{1}{C_z \|A\|}$, hence a bounded linear operator. (Remember that the space of bounded linear operators from E to F is a Banach space.)

It follows by an easy computation that S is the two-sided inverse of Az' + B.

q.e.d.

(C 3)

Let k be a 2-disk in C (differentiable) such that for ∂k (the boundary of k) $\partial k \cap \mathrm{Spec}(A, B) = \emptyset$ holds.

We define

$$Q_k^F (A, B) = \frac{1}{2\pi i} \int_{\partial k} A (Az + B)^{-1} dz \quad .$$

This integral is well defined, since for every $z \in \partial k$ $(Az + B)^{-1}$ is well defined and bounded. For every $z \in \partial k$ $A(Az + B)^{-1}$ is a linear map of F into itself, the integral is approximated (uniformly) by linear combinations of these linear maps, and therefore $Q_k^F(A, B)$ is a bounded linear operator in F. (∂k is compact therefore $\|A\| . \|Az + B)^{-1}\| < C$ for a certain C>0).

Similarly we define

$$Q_k^E(A, B) = \frac{1}{2\pi i} \int_{\partial k} (Az + B)^{-1} A \, dz$$

$Q_k^E(A, B) \in \mathrm{End}\, E$ is a linear operator on E .

(C. 4) Lemma

If k and k' are two disks such that

$$\partial k \cap \partial k' = \emptyset$$
$$\partial k \cap \mathrm{Spec}(A, B) = \emptyset$$
$$\partial k' \cap \mathrm{Spec}(A, B) = \emptyset$$

then the following holds

a) if $k \cap k' = \emptyset$ $\quad Q_k^E(A, B)\, Q_{k'}^E(A, B) = 0 = Q_{k'}^E(A, B)\, Q_k^E(A, B)$

b) if $k' \subset k$ $\quad Q_k^E(A, B)\, Q_{k'}^E(A, B) = Q_{k'}^E(A, B) = Q_{k'}^E(A, B)\, Q_k^E(A, B)$.

Proof: $Q_k^E(A, B)\, Q_{k'}^E(A, B) = (\frac{1}{2\pi i})^2 \int_{\partial k} (A\xi+B)^{-1}A\, d\xi \int_{\partial k'} (A\eta+B)^{-1}A d\eta$

$$= (\frac{1}{2\pi i})^2 \iint (A\xi+B)^{-1} A(A\eta+B)^{-1}A\, d\xi\, d\eta .$$

Now $(A\xi+B)^{-1}A(A\eta+B)^{-1}A = \dfrac{(A\xi+B)^{-1}A - (A\eta+B)^{-1}A}{\eta-\xi}$

For $\dfrac{(A\xi+B)^{-1} - (A\eta+B)^{-1}}{\eta-\xi} = \dfrac{(A\xi+B)^{-1}(A\eta+B)(A\eta+B)^{-1} - (A\xi+B)^{-1}(A\xi+B)(A\eta+B)^{-1}}{\eta-\xi}$

$$= (A\xi+B)^{-1}A(A\eta+B)^{-1} .$$

So the integral becomes $(\frac{1}{2\pi i})^2 \iint \dfrac{(A\xi+B)^{-1}A - (A\eta+B)^{-1}A}{\eta-\xi}$

This integral is well defined for $|\eta-\xi|$ stays larger than a certain $\varepsilon > 0$ since ∂k and $\partial k'$ are disjoint and compact.

Let now $k \cap k' = \emptyset$.

Then $\frac{1}{2\pi i} \int_{\partial k'} \dfrac{(A\xi+B)^{-1}A}{\eta-\xi} d\eta = \frac{1}{2\pi i} (A\xi+B)^{-1}A \int_{\partial k'} \frac{1}{\eta-\xi} d\eta = 0$

for every $\xi \in \partial k$, and

$$\frac{1}{2\pi i} \int_k \frac{(A\eta+B)^{-1}A}{\eta-\xi} d\xi = 0 \quad \text{for every} \quad \eta \in \partial k' .$$

So $(\frac{1}{2\pi i})^2 \iint \dfrac{(A\xi+B)^{-1}A - (A\eta+B)^{-1}A}{\eta-\xi} d\eta d\xi = 0$.

Suppose $k' \subset k$.

Then $\frac{1}{2\pi i} \int_{\partial k'} \dfrac{(A\xi+B)^{-1}A}{\eta-\xi} d\eta = 0$ and $\frac{1}{2\pi i} \int_{\partial k} \dfrac{(A\eta+B)^{-1}A}{\eta-\xi} d\xi =$

$= \frac{1}{2\pi i} (A\eta+B)^{-1}A \int_{\partial k} \frac{d\xi}{\eta-\xi} = -(A\eta+B)^{-1}A$ for every $\eta \in \partial k'$

(η lies inside the integration circle).

So

$$(\frac{1}{2\pi i})^2 \iint \frac{(A\xi+B)^{-1}A - (A\eta+B)^{-1}A}{\eta-\xi} d\eta d\xi = \frac{1}{2\pi i} \int_{\partial k'} (A\eta+B)^{-1}A\, d\eta = Q_{k'}^E(A, B) .$$

q.e.d

(C. 5) Proposition

$Q_k^E(A,B)$ is a projection

(i.e $Q_k^E(A, B)\, Q_k^E(A, B) = Q_k^E(A, B))$.

Proof: $\mathrm{Spec}(A, B) \subset \mathbb{C}$ is closed in $\mathbb{C}$, and ∂k is compact and $\partial k \cap \mathrm{Spec}(A,B) = \emptyset$, so we can find $\varepsilon>0$ such that if

$$U = \{z' \mid \exists\, z \in \partial k \text{ with } |z'-z| < \varepsilon\}$$

then $U \cap \mathrm{Spec}(A, B) = \emptyset$.

So we can find a disk $k' \subset k$ such that $k' \subset k^o$ and $\partial k' \subset U$.

By the theorem of Cauchy then

$$\int_{\partial k'} (Az + B)^{-1} dz = \int_{\partial k} (Az + B)^{-1} A\, dz \quad .$$

Applying (C.4) we get the desired result.

One proves in the same way that $Q_k^F(A, B)$ is a projection

(C.6) Lemma

Let $\xi \in \mathbb{C}$. If $\xi \notin \mathrm{Spec}(A, B)$, then $A(A\xi + B)^{-1}B = B(A\xi+B)^{-1}A$.

Proof if $\xi = 0$ trivial, suppose $\xi \neq 0$

$\xi A(A\xi + B)^{-1}B - B(A\xi + B)^{-1}A\xi = (B+\xi A)(A\xi + B)^{-1}B - B(A\xi + B)^{-1}B - B(A\xi + B)^{-1}(A\xi+B) +$

$+ B(A\xi + B)^{-1}B = B-B = 0$. q.e.d.

(C. 7) Proposition

The Banach space E splits as a direct sum of the linear space

$E_+ = Q_k^E(A, B)(E)$ and $E_- = (I - Q_k^E(A, B))(E)$, F splits in the same way. Now $(Az + Bw)(E_+) \subset F_+$ and $(Az + Bw)(E_-) \subset F_-$, so $(Az + Bw)$ splits into a direct sum of two linear maps $(Az + Bw)_+ : E_+ \to F_+$, $(Az + Bw)_- : E_- \longrightarrow F_-$,

$(Az + Bw) = (Az + Bw)_+ \oplus (Az + Bw)_-$.

Proof: The following diagram is commutative

$$\begin{array}{ccc} E & \xrightarrow{\;Az + Bw\;} & F \\ {\scriptstyle Q_k^E(A,\,B)}\downarrow & & \downarrow{\scriptstyle Q_k^F(A,\,B)} \\ E & \xrightarrow{\;Az + Bw\;} & F \end{array}$$

For $A(A\xi + B)^{-1}(Az + Bw) = (Az + Bw)(A\xi + B)^{-1}A$ by (C.6)

so $\left(\int_{\partial k} A(A\xi + B)^{-1} d\xi \right) (Az + B) = (Az + B) \int_{\partial k} (A\xi + B)^{-1} A\, d\xi$

The $Q_k^E(A, B)$ and $Q_k^F(A, B)$ being projections, the spaces E and F split as indicated in the proposition. The commutativity of the diagram gives immediately that $(Az + Bw)(E_+) \subset F_+$ and $(Az + Bw)(E_-) \subset F_-$ and therefore $(Az + Bw)$ splits into $(Az + Bw)_+ \oplus (Az + Bw)_-$ ($(Az + Bw)_+$ being $Az + Bw$ restricted to E_+ analogously $(Az + Bw)_-$)

(C. 8)

Proposition:

Let z, w be complex numbers, not both zero. If $pr(z, w) \notin k$, then $(az + Bw)_+$ is invertibel with bounded inverse .

Proof: Consider $(Az + Bw) \circ \frac{1}{2\pi i} \int_{\partial k} \frac{(A\xi + B)^{-1}}{w\xi - z} d\xi$.

∂k is compact. So there is an $\varepsilon > 0$ such that $|w\xi - z| > \varepsilon$ for all $\xi \in \partial k$.

$\|(A\xi + B)^{-1}\|$ is bounded, since ∂k is compact and $\mathrm{Spec}(A, B) \cap \partial k = \emptyset$. Thus the integral is well-defined. Now for $w\xi \neq z$

$$\frac{(Az + Bw)(A\xi + B)^{-1}}{w\xi - z} = \frac{w}{w\xi - z} \cdot \mathrm{Id} - A(A\xi + B)^{-1} .$$

So

$$(Az + Bw) \frac{1}{2\pi i} \int_{\partial k} \frac{(A\xi + B)^{-1}}{w\xi - z} d\xi = \begin{cases} \mathrm{Id} - Q_k^F(A, B), & \text{if } pr(z,w) \in k^\circ \\ - Q_k^F(A, B), & \text{if } pr(z,w) \notin k \end{cases}$$

So $- \frac{1}{2\pi i} \int_{\partial k} \frac{(A\xi + B)^{-1}}{w\xi - z} d\xi \Big|_{F_+}$ is a right inverse of $(Az + Bw)_+$. Similarly one checks that $- \frac{1}{2\pi i} \int_{\partial k} \frac{(A\xi + B)^{-1}}{w\xi - z} d\xi \Big|_{F_+}$ is a left inverse of $(Az + Bw)_+$

q.e.d.

Remark: Taking $w = 0$, we see that in particular A_+ is invertible with bounded inverse.

Exercise: Prove that if $pr(z, w) \in k^\circ$, then $(Az + Bw)_-$ is invertible with bounded inverse.

On the proof of Bott's periodicity theorem.

(C.9) We define the "Hopf bundle" H over S^2 to be the dual of the universal one bundle $E_{1,1}$ over $G_{1,1} = P_1(\mathbb{C}) = S^2$. For definition see A.24 . Let X be a compact space, $\pi_1 : X \times S^2 \to X$, $\pi_2 : X \times S^2 \to S^2$ the projections. Then the periodicity theorem says:

Periodicity Theorem:

The homomorphism

$$f : K(X) \otimes_{\mathbb{Z}} K(S^2) \longrightarrow K(X \times S^2)$$

given by $\qquad x \otimes y \longmapsto \pi_1^*(x) \cdot \pi_2^*(y)$

is a ring isomorphism.

(C.10) We deduce this from the following two propositions:

Proposition 1: If x is an arbitrary element of $K(X \times S^2)$, then there exist $x_i \in K(X)$, $i = 1,2$, such that

$$x = \pi_1^* x_1 + \pi_1^* x_2 \cdot \pi_2^*([H]-1).$$

Moreover x_1 and x_2 are uniquely determined by x.

Proposition 2: In $K(S^2)$ holds

$$([H]-1)^2 = 0.$$

Proposition 1 states that $K(X \times S^2) \simeq K(X) \oplus K(X)$ as an abelian group. For $X = \{point\}$ we see that $K(S^2)$ is the free abelian group with 1 and $[H]$ as generators. Proposition 2 determines $K(S^2)$ as a ring and thereby the ring structure of $K(X \times S^2)$. Now the theorem is an easy consequence:
If $x = \pi_1^* x_1 + \pi_1^* x_2 \cdot \pi_2^*([H] - 1)$, then $g : K(X \times S^2) \longrightarrow K(X) \otimes_{\mathbb{Z}} K(S^2)$, defined by $g(x) = (x_1 - x_2) \otimes 1 + x_2 \otimes [H]$, is an isomorphism of rings and f is its inverse.

(C.11) We shall prove Propositions 1 and 2 except the uniqueness of x_2. The proof is essentially based on clutching and the fact that homotopic clutching functions give rise to isomorphic vectorbundles.

First observe that if $p \in S^2$ and $i_1 : X \to X \times S^2$ is given by $i_1(x) = (x, p)$, then $\pi_1 i_1 = \mathrm{id}$ and $\pi_2 i_1 = \mathrm{const}$. Hence $i_1^* x = x_1$, because $i_1^* \pi_2^* ([H] - 1) = 0$. Thus we have the uniqueness of x_1. It is sufficient to prove the existence of x_1, x_2 for $x = [E]$, where E is a vectorbundle over $X \times S^2$.

Consider S^2 as the union of the two disks

$$D_0 = \{z \mid z \in \mathbb{C} \wedge |z| \leq 1\} \text{ and } D_\infty = \{z \mid z \in \mathbb{C} \wedge |z| \geq 1\} \cup \{\infty\} .$$

$D_0 \cap D_\infty = S^1$. We define the following maps:

$$\pi_0 = \pi_1 | X \times D_0 , \qquad \pi_\infty = \pi_1 | X \times D_\infty , \qquad \pi = \pi_1 | X \times S^1$$

$$S_0 : X \to X \times S^2 , \qquad S_0(x) = (x, 0)$$

$$S_\infty : X \to X \times S^2 , \qquad S_\infty(X) = (x, \infty) .$$

Let $E_0 = s_0^* E$, $E_\infty = s_\infty^* E$. π_0 and π_∞ are homotopy equivalences. Therefore $\pi_0^* E_0 = E|X \times D_0$, $\pi_\infty^* E_\infty = E|X \times D_\infty$. Since s_0 and s_∞ are homotopic, E_0 is isomorphic to E_∞. We identify the two bundles and use the notation F for both. It is clear that E can be obtained by clutching $\pi_0^* E_0 = \pi_0^* F$ and $\pi_\infty^* E_\infty = \pi_\infty^* F$, the clutching function being the identity of $E|X \times S^1$. So we have the following result: Any vectorbundle over $X \times S^2$ can be obtained by clutching $\pi_0^* F$ and $\pi_\infty^* F$, where F is some vectorbundle over X and the clutching function ϕ is some isomorphism of $\pi^* F$ onto itself. ϕ may be considered as a continuous map

$$\phi : X \times S^1 \longrightarrow \mathrm{Hom}(F, F)$$

such that $\phi(x, z) \in \mathrm{Iso}(F_x, F_x)$.

For X = {point} this means that each line bundle over S^2 is given by a map $\phi: S^1 \to GL(1, \mathbb{C}) = \mathbb{C} - \{0\}$. A clutching function which gives rise to the bundle H^*, is the map which assigns to $z \in S^1$ the isomorphism $z \cdot \mathrm{Id}_{\mathbb{C}}$. We simply denote it by "z". Now we prove $H^2 \oplus 1 \simeq H \oplus H$ which implies Proposition 2. According to Proposition A. 17, $H^2 \oplus 1$ and $H \oplus H$ are given by the clutching matrices $\begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$, respectively. But these are homotopic in $GL(2,\mathbb{C})$, since $\begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$.

Now we return to the general case of a vectorbundle E over $X \times S^2$. Let F be a vectorbundle over X such that $E \simeq \pi_0^* F \underset{\phi}{\cup} \pi_\infty^* F$, where ϕ is some clutching function as above. Choose a Hermitian metric on F. Thus F_x is a finite-dimensional normed space for all $x \in X$.

<u>1 st step</u>: The k-th Fourier-coefficient of ϕ, $a_k(x) := \frac{1}{2\pi i} \int_{S^1} \frac{\phi(x,z)}{z^{k+1}} dz$, is an endomorphism of F_x and depends continuously on x. Define $S_N(x,z) = \sum_{k=-N}^{N} a_k(x) z^k \in \mathrm{End}\, F_x$. By the theorem of Fejér the sequence

$$\phi_N = \frac{1}{N}(S_0 + \ldots + S_{N-1})$$

converges to ϕ uniformly in z for each $x \in X$. As X is compact, ϕ_N converges uniformly even on all of $X \times S^1$. The ϕ_N need not be clutching functions, because non-singularity is not guaranteed. But as the clutching functions form an open set, ϕ_N is a clutching function homotopic to ϕ for smome N. So it is sufficient to prove the existence of the x_i's only for clutching function which are finite Laurent series in z as is ϕ_N.

<u>2nd step</u>: Let E be given by a clutching function $\phi(x,z) = \sum_{k=-m}^{m} f_k(x) z^k$.

Then $E \otimes \pi_2^*(H^*)^m$ is obtained by $\phi \cdot z^m$. Now assume $[E \otimes \pi_2^*(H^*)^m] =$

$= \pi_1^* x_1 + \pi_1^* x_2 \cdot \pi_2^*([H]-1)$.

Then $[E] = \pi_1^* x_1 \cdot \pi_2^*[H]^m + \pi_1^* x_2 \cdot \pi_2^*(([H]-1)[H]^m)$.

$[H]^m = 1+m([H]-1)$ by Proposition 2. Thus
$[E] = \pi_1^* x_1 + \pi_1^*(mx_1+x_2)\ \pi_2^*([H]-1)$. But $\phi \cdot z^m$ is a polynomial in z. So we may confine ourselves to polynomial clutching functions.

3rd step: Let p be a polynomial clutching function, $p(x,z) = \sum_{i=0}^{m} f_i(x)\, z^i$. Clearly it is sufficient to verify the existence of the x_i's for the vectorbundle.

$$(\pi_0^* F \underset{p}{\cup} \pi_\infty^* F) \oplus m \cdot \pi_1^* F\ ,$$

which is given by the clutching matrix $\begin{pmatrix} p_1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ of m+1 rows. We assert:

$$\begin{pmatrix} p_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \simeq \begin{pmatrix} f_0 & \cdots & \cdots & f_m \\ -z & 1 & & \\ & \ddots & \ddots & \\ & & -z & 1 \end{pmatrix}$$

For if $p_1, \ldots, p_m$ are suitable polynomals, then

$$\begin{pmatrix} 1 & p_1 & \cdots & p_m \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} p_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ -z & 1 & & \\ & \ddots & \ddots & \\ & & -z & 1 \end{pmatrix} = \begin{pmatrix} f_0 & \cdots & & f_m \\ -z & 1 & & \\ & \ddots & \ddots & \\ & & -z & 1 \end{pmatrix},$$

the first and the third factor being homotopic to the identity.

4th step: We only have to deal with a vectorbundle E coming from a linear clutching function $L(x,z) = A(x)\, z + B(x)$, where $A(x)$ and $B(x)$ are automorphisms of F_x depending continuously on x. We apply C.7 . D_0 is our disk k in $\mathbb{C}$. Then $\partial k \cap \operatorname{Spec}(A(x), B(x)) = \emptyset$ is satisfied, because $A(x)z + B(x)$ is a clutching function for $z \in S^1$. For each $x \in X$, $(\xi, \omega) \in C_2 - \{0\}$, $A(x)\xi + B(x)\omega$ splits into a direct sum of linear maps

$$A_+(x)\xi + B_+(x)\omega\colon\ F_x^+ \longrightarrow \tilde{F}_x^+ \qquad \text{and}$$
$$A_-(x)\xi + B_-(x)\omega\colon\ F_x^- \longrightarrow \tilde{F}_x^-$$

such that $F_x^+ \oplus F_x^- = \tilde{F}_x^+ \oplus \tilde{F}_x^- = F_x$. As F_x^+ is the range of a projection operator depending continuously on x, namely $Q_{D_0}^{F_x}(A_{(x)}, B_{(x)})$, $F^+ = \bigcup_{x \in X} F_x^+$ is a subbundle of F. (Note that for a projection operator rank = trace, and trace is a continuous function.) Similarly define the subbundles F^-, $\tilde{F}^+$, $\tilde{F}^-$. Then $F = F^+ \oplus F^- = \tilde{F}^+ \oplus \tilde{F}^-$.

By C.8 $A_+\xi + B_+\omega$ and $A_-\xi + B_-\omega$ are bijective (hence clutching functions) for $pr(\xi,\omega) \in D_\infty$ and $pr(\xi,\omega) \in D_0$ respectively. (pr: $\mathbb{C}_2 - \{0\} \to P_1\mathbb{C} = S^2$ the canonical projection.) Especially A_+ is bijective ($\omega = 0$), and so is $B_-(\xi = 0)$. Therefore $A_+z + B_+$ is homotopic (as a clutching function!) to A_+z and $A_-z + B_-$ to B_-. Homotopies are given by $A_+z + tB_+$ and $tA_- z + B_-$ respectively So it does not matter whether we clutch

by L or by $\begin{pmatrix} A_+z & 0 \\ 0 & B_- \end{pmatrix}$.

$$E \simeq (\pi_0^* F^+ \underset{A_+z}{\smile} \pi_\infty^* \tilde{F}^+) \oplus (\pi_0^* F^- \underset{B_-}{\smile} \pi_\infty^* \tilde{F}^-)$$

$$\simeq (\pi_1^*(F^+ \underset{A_+}{\smile} \tilde{F}^+) \otimes \pi_2^* H^*) \oplus \pi_1^*(F^- \underset{B_-}{\smile} \tilde{F}^-)$$

Hence in $K(X \times S^2)$:

$$[E] = \pi_1^*[F^+ \underset{A_+}{\smile} \tilde{F}^+] . \pi_2^* [H^*] + \pi_1^*[F^- \underset{B_-}{\smile} F^-]$$

Now it is easy to determine x_1, x_2 such that

$[E] = \pi_1^* x_1 + \pi_1^* x_2 . \pi_2^*([H] - 1)$.

(Da $H^* = [H] - [1] - [1]$ und Relation (2))
(und $H^* \otimes H \sim 1$)

Remark: The uniqueness of x_2 is obtained by a thorougher investigation of the process of simplifying the clutching function.

D. Exact Sequences in K-Theory

(D.1) We recall the definition of K' : If $f : X \longrightarrow$ point is the projection of a topological space X onto the space "point" consisting of one point, then by definition $K'(X) = \mathrm{Coker}\, K(f)$. Thus we have an exact sequence

$$0 \longrightarrow K(\text{point}) \longrightarrow K(X) \longrightarrow K'(X) \longrightarrow 0$$

which splits by B.8. It is clear that for a continuous map $g: X \to Y$, $K(g)$ induces a homomorphism $K'(g) : K'(Y) \to K'(X)$. So K' is a contravariant functor from the category of topological spaces and continuous maps to the category of abelian groups and homomorphisms.

(D.2) Let (Y, X) be a compact pair. As Y/X has a distinguished base-point, $\tilde{K}(Y/X)$ and $K'(Y/X)$ are canonically isomorphic.

Proposition: Let $i : X \to Y$ be the inclusion, $p : Y \to Y/X$ the projection. Then the sequence

$$\tilde{K}(Y/X) \xrightarrow{K(p)|\cdots} K(Y) \xrightarrow{K(i)} K(X)$$

is exact.
(Remember that $\tilde{K}(Y/X) \subset K(Y/X)$, if B.8).

Proof. If $a \in \tilde{K}(Y/X)$, then by B.5 $a = [E] - n.[1]$ for some vector bundle E over Y/X and some natural number n. $K(i)\, K(p)a = K(pi)a = [(pi)^*E] - n.[1] = (d-n)[1]$ where $d = \dim (E|\text{ base-point})$. But $d = n$ because $a \in \mathrm{Ker}(K(Y/X) \to K(\text{base-point}))$. Thus the composition of the two homomorphisms of the sequence is zero.
Now let $[F] - m.|1_Y| \in K(Y)$ such that $K(i)([F] - m[1_Y]) = 0$, i.e. $[F|X] = m[1_X]$ in K(X). By B.7, $F|X \oplus k.1_X \simeq (m+k).1_X$ for some natural number k, whence $\dim F|X = m$. So there exists a trivialisation α of $G = F \oplus k.1_Y$ over X.
As $G \simeq p^*G_{(\alpha)}$, $[F] - m[1_Y] = K(p)(G_{(\alpha)} - (k+m)[1])$. We have to show:
$[G_{(\alpha)}] - (k+m)[1] \in \tilde{K}(Y/X) = \mathrm{Ker}(K(Y/X) \longrightarrow K(\text{base-point}))$.
This is clear, since $\dim F|X = m$. q.e.d.

We now extend the definition of the function K to the category of compact pairs. If (Y, X) is a compact pair, let $K(Y, X) = K'(Y/X)$. For maps of compa ct pairs, K is defined in the obvious way. If we put $Y/\emptyset = Y \cup$ point (disjoint union), this extends the former definition of K, because $K(Y, \emptyset) = K'(Y \cup \text{point}) = K(Y)$.
Define $K(\emptyset) = 0$. Then for every compact pair (Y, X), we have an exact sequence of abelian groups

$$K(Y, X) \longrightarrow K(Y) \longrightarrow K(X).$$

(D.3) Mapping Cylinder, Mapping Cone.

Until D.8 we shall assume all spaces occurring nonempty.

Let X, Y be compact spaces, $f : X \to Y$ continuous. The mapping cylinder Z_f of f is defined to be a certain quotient space of the disjoint union $X \times I \cup Y$. If $g : X \times \{0\} \to Y$ is the map $g(x, 0) = f(x)$, define $Z_f = X \times I \cup_g Y$. There is a natural continuous inclusion $i : Y \to Z_f$ induced by the projection $p : X \times I \cup Y \to Z_f$. The natural injection $j : X \to X \times \{1\}$ yields an injection $p \circ j : X \to Z_f$ which is also continuous. So we may consider X and Y as subspaces of Z_f. The mapping cone C_f of f is defined as Z_f/X. Both Z_f and C_f are compact. The mapping cone of id_X is also called "the cone over X", notation CX. If f is injective, then $CX \subset C_f$.

(D.4)

If X and Y are compact spaces, a map $f : X \to Y$ gives rise to the following sequence of spaces and maps (the so-called Puppe sequence):

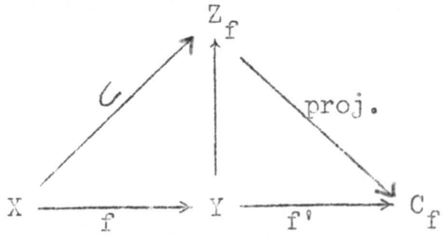
$$X \xrightarrow{f} Y \xrightarrow{f'} C_f \xrightarrow{f''} C_{f'} \xrightarrow{f'''} C_{f''}$$

Proposition: The sequence obtained by applying K' to the Puppe sequence is exact.

Proof: Consider three consecutive spaces of the Puppe sequence. The third space is the mapping cone of the map from the first one to the second one. Therefore it is sufficient to prove exactness of the sequence

$$K'(C_f) \longrightarrow K'(Y) \longrightarrow K'(X).$$

The diagramm



is commutative up to homotopy. So it induces a commutative one for K'. As the inclusion $Y \subset Z_f$ is a homotopy equivalence, we only need to show that

$$K'(C_f) \longrightarrow K'(Z_f) \longrightarrow K'(X)$$

is exact. This is clear by D.2 and B.8 (choose a base-point in X). q.e.d.

(D.5)

The suspension ΣX of a compact space X is the quotient of $X \times I$ by the following equivalence relation $\sim$: If $x, y \in X \times I$, define $x \sim y$ if and only if $x, y \in X \times \{0\}$ or $x, y \in X \times \{1\}$. Then $\Sigma X = X \times I/\sim$.

The suspension of a map $f : X \to Y$ is the function $\Sigma f : \Sigma X \to \Sigma Y$ which makes the following diagram commutative:

$$\begin{array}{ccc} X \times I & \xrightarrow{f \times id} & Y \times I \\ {\scriptstyle proj.}\downarrow & & \downarrow{\scriptstyle proj.} \\ \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y \end{array}$$

Σf is continuous. Thus Σ is a functor from the compact category to itself. If f and g are homotopic maps, then Σf and Σg are homotopic.

We are now in a position to define a diagram which will turn out to be commutative up to homotopy. Let (Y, X) be a compact pair, f the inclusion

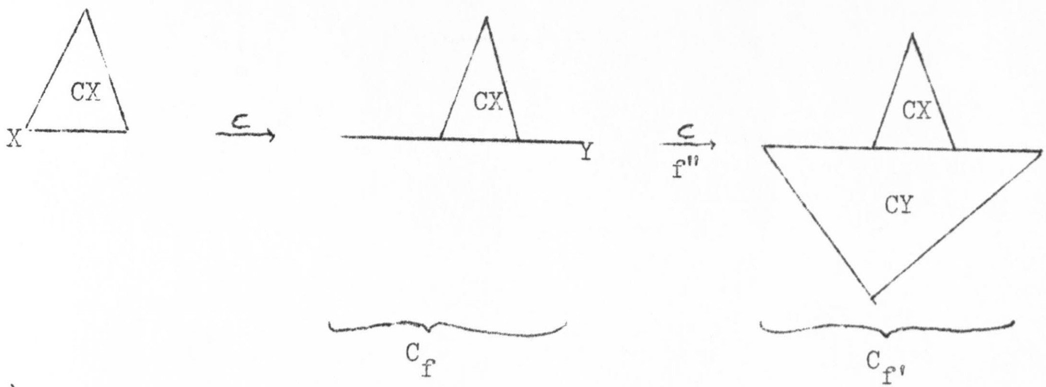
(*)

$$\begin{array}{ccccccccccccccccc} X & \xrightarrow{f} & Y & \xrightarrow{f'} & C_f & \xrightarrow{f''} & C_{f'} & \xrightarrow{f'''} & C_{f''} & \xrightarrow{f^{(4)}} & C_{f'''} & \xrightarrow{f^{(5)}} & C_{f^{(4)}} & \longrightarrow & C_{f^{(5)}} & \longrightarrow & \cdots \\ & & & & & {\scriptstyle h}\searrow & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle g'''} & & \downarrow & & \\ & & & & & & \Sigma X & \longrightarrow & \Sigma Y & \longrightarrow & \Sigma C_f & \longrightarrow & \Sigma C_{f'} & \longrightarrow & \Sigma C_{f''} & \longrightarrow & \cdots \\ & & & & & & & & & & & {\scriptstyle \Sigma h}\searrow & \downarrow{\scriptstyle \Sigma g} & & \downarrow{\scriptstyle \Sigma g'} & & \\ & & & & & & & & & & & & \Sigma^2 X & \longrightarrow & \Sigma^2 Y & \longrightarrow & \cdots \end{array}$$

The upper row is the Puppe sequence of f. The n-th row is the suspension of the (n-1)-th row, suitably shifted. g is defined as follows: In the diagram

$$\begin{array}{ccccc} CX & \longrightarrow & C_f & \xrightarrow{f''} & C_{f'} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ CX/X & \xrightarrow{\phi} & C_f/Y & \xrightarrow{\psi} & C_{f'}/CY \end{array}$$

the vertical arrows stand for natural projections, the upper horizontal arrows for inclusion maps, and ϕ, ψ are defined as functions such that the diagram is commutative. ϕ and ψ are bijective. ϕ and ψ are continuous, because the vertical maps are onto and all spaces occurring in the diagram are compact.
Thus ϕ and ψ are homeomorphisms. Define $g = \phi^{-1} \psi^{-1} \pi$.



$g^{(n)}$ is defined in the same way as g, namely as the composition of the maps

$$C_{f^{(n+1)}} \longrightarrow C_{f^{(n+1)}}/C(C_{f^{(n-1)}}) \xrightarrow{\sim} C_{f^{(n)}}/C_{f^{(n-1)}} \xrightarrow{\approx} \Sigma C_{f^{(n-2)}} .$$

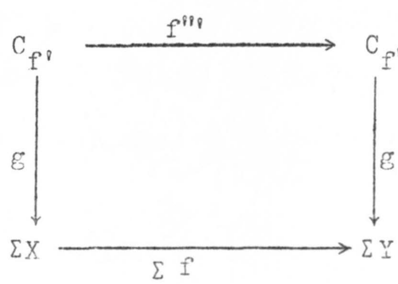
(Put $C_f = C_{f^{(0)}}$, $Y = C_{f^{(-1)}}$.)

Finally put $h = gf''$. Now the diagram (*) is well-defined.

(D.6)

Proposition: The diagram (*) is commutative up to homotopy.

Proof: Clearly, it is sufficient to verify that $g' \circ f'''$ and $\Sigma f \circ g$ are homotopic.



Consider the space $T = X \times [0,1] \cup Y \times [-1,0]$ as a subspace of $Y \times [-1, 1]$. $C_{f'}$ is a quotient space of T : Take $X \times \{1\}$ and $Y \times \{-1\}$ into one point each. Let $p : T \to C_{f'}$ be the natural projection. Define $F : T \times I \to Y \times I$ by

$$F(z, t, s) = \begin{cases} (z, 1) & \text{if} \quad 1-s \leq t \\ (z, t+s) & \text{if} \quad -s \leq t \leq 1-s \\ (z, 0) & \text{if} \quad t \leq -s \end{cases}$$

F is continuous and induces a function $G : C_{f'} \times I \to \Sigma Y$ which is defined by requiring commutativity of the diagram

$$\begin{array}{ccc}
T \times I & \xrightarrow{\quad F \quad} & Y \times I \\
{\scriptstyle p \times id}\downarrow & & \downarrow{\scriptstyle nat.proj.} \\
C_{f'} \times I & \xrightarrow[\quad G \quad]{} & \Sigma Y
\end{array}$$

The usual compactness argument shows that G is continuous. Now check $G_0 = \Sigma f \circ g$ and $G_1 = g' \circ f'''$. q.e.d.

(D.7)

The Puppe diagram is the following diagram induced by (*):

$$(**)\qquad \begin{array}{ccccccccccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{f'} & C_f & \xrightarrow{f''} & C_{f'} & \xrightarrow{f'''} & C_{f''} & \xrightarrow{f^{(4)}} & C_{f'''} & \longrightarrow & C_{f^{(4)}} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} & & \downarrow & & \downarrow & & \\
X & \xrightarrow[f]{} & Y & \xrightarrow[f']{} & C_f & \xrightarrow[h]{} & \Sigma X & \xrightarrow[\Sigma f]{} & \Sigma Y & \xrightarrow[\Sigma f']{} & \Sigma C_f & \xrightarrow[\Sigma h]{} & \Sigma^2 X & \longrightarrow & \cdots
\end{array}$$

The vertical maps are compositions of vertical maps of (*). So to prove that the vertical maps of (**) induce isomorphisms for K', it is sufficient to show that $K'(\Sigma^n g)$ is an isomorphism for all $n=0,1,\ldots$.

This follows from

<u>Lemma</u>: Let (A, Z) be a compact pair, Z contractible to a point, $p : A \to A/Z$ the projection map. Then $K'(\Sigma^n p)$ is bijective.

Proof: Consider the diagram

$$\begin{array}{ccc}
\Sigma^n A & \xrightarrow{\quad \Sigma^n p \quad} & \Sigma^n(A/Z) \\
& {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle \nu} & \\
& \Sigma^n A/\Sigma^n Z &
\end{array}$$

q is the natural projection, ν is uniquely defined by requiring $\nu \circ \Sigma^n p = q$. Then ν is continuous, because A and Z are compact. $\nu^{-1}(\Sigma^n Z) = \Sigma^n(\text{point})$. By A.21 q and ν induce isomorphisms for K'. Hence the proposition.

Now remember the definition of g in D.6: $g = \phi^{-1} \psi^{-1} \pi$.

As CY is contractible, $K'(\Sigma^n \pi)$ is an isomorphism by our lemma. Therefore $K'(\Sigma^n g)$ is bijective. As the Puppe diagram (**) is commutative up to homotopy, we get a commutative diagram, if we apply K' to it. Using D.4, we have the result:

<u>Theorem</u>: The lower row of (**) induces an exact sequence for K':

$$K'(X) \longleftarrow K'(Y) \longleftarrow K'(C_f) \longleftarrow K'(\Sigma X) \longleftarrow K'(\Sigma Y) \longleftarrow K'(\Sigma C_f) \longleftarrow K'(\Sigma^2 X) \longleftarrow \cdots$$

(D.8)

We want to establish an exact sequence, replacing C_f by Y/X. Define $\phi : Y/X \to C_f/CX$ such that the diagram

$$\begin{array}{ccc} Y & \longrightarrow & C_f \\ \text{proj.}\downarrow & & \downarrow\text{proj.} \\ Y/X & \xrightarrow[\phi]{} & C_f/CX \end{array}$$

is commutative. Then ϕ is a homeomorphism (cf. D.5). So we have a map $v : C_f \longrightarrow Y/X$ such that $K'(\Sigma^n v)$ is bijective (apply D.7. Lemma). Using v we may replace C_f by Y/X in the sequence of the last theorem to get an exact sequence

$$K'(X) \leftarrow K'(Y) \leftarrow K'(Y/X) \leftarrow K'(\Sigma X) \leftarrow K'(\Sigma Y) \leftarrow K'(\Sigma(Y/X)) \leftarrow K'(\Sigma^2 X) \leftarrow \cdots$$

Remark 1: The homomorphisms $K'(\Sigma^n(Y/X)) \to K'(\Sigma^n Y)$ are induced by the natural projection $Y \to Y/X$.

Remark 2: The above sequence is natural with respect to a continuous map of compact pairs $(Y, X) \longrightarrow (B, A)$.

(D.9) Definition and Properties of K-Theory.

From now on we admit the empty space again.
For a topological space Y let Y^+ denote the disjoint union of Y and a point. A map $g : X \to Y$ induces a map $g^+ : X^+ \to Y^+$, homotopic maps from X to Y induce homotopic maps from X^+ to Y^+. If $X \subset Y$, it is convenient to define $Y/X = Y^+/X^+$. Then a quotient space Y/X is never empty and has always a distinguished base-point.

Using the functor K', we now define a cohomology theory - the so-called K-theory - on the category of compact pairs and their maps which satisfies all the axioms of Eilenberg and Steenrod except the dimension axiom. The latter does not hold because it is not compatible with the Bott periodicity, as is seen lateron.

<u>Definition</u>: For $N = 0,1,2,\ldots$ and any compact pair (Y, X), put

$$K^{-n}(Y, X) = K'(\Sigma^n(Y/X)).$$

Remark: The righthand side is isomorphic to $\tilde{K}(\Sigma^n(Y/X))$. For $K^{-n}(Y, \emptyset)$ we write $K^{-n}(Y)$. $K^0(Y) = K'(Y^+/\emptyset^+) = K'(Y^+) = K(Y)$. Clearly K^{-n} is a contravariant functor, if we define K^{-n} for maps in the obvious way. We define the exact sequence.

$$K^0(X) \leftarrow K^0(Y) \leftarrow K^0(Y, X) \leftarrow K^{-1}(X) \leftarrow K^{-1}(Y) \leftarrow K^{-1}(Y, X) \leftarrow K^{-2}(X) \leftarrow \cdots$$

Clearly K^{-n} is a contravariant functor, if we define K^{-n} for maps in the obvious way. We define the exact sequence

$$K^{o}(X) \leftarrow K^{o}(Y) \leftarrow K^{o}(Y, X) \leftarrow K^{-1}(X) \leftarrow K^{-1}(Y) \leftarrow K^{-1}(Y, X) \leftarrow K^{-2}(X) \leftarrow \dots$$

of the pair (Y, X) as the exact sequence of D.8 for (Y^{+}, X^{+}) instead of (Y, X). (Note that $K^{-n}(Y) = K'(\Sigma^{n}(Y^{+}/\emptyset^{+})) = K'(\Sigma^{n}(Y^{+}))$.) Thereby we have already defined the boundary operator $\delta : K^{-n-1}(X) \longrightarrow K^{-n}(Y, X)$. By remark 2 of D.8, δ is compatible with maps. The homotopy axiom is satisfied, because two homotopic maps induce the same homomorphism for K as well as for K'. The excision axiom holds in this formulation:

Proposition: Let (Y, X) be a compact pair, $A \subset X$, A open in Y. Then the inclusion $i : (Y-A, X-A) \rightarrow (Y, X)$ induces isomorphisms for K^{-n}.

Remark: Realise that Y-A and X-A are compact if and only if A is open in Y. Furthermore $\tilde{K}^{n}(Y, X)$ and $K^{-n}(Y/X, \text{point})$ are isomorphic by definition and an isomorphism is induced by the natural map $(Y, X) \rightarrow (Y/X, \text{point})$.
Proof of the proposition: $X = \emptyset$ is trivial. So let $X \neq \emptyset$. Assume $A \neq X$. In the commutative diagram

$$\begin{array}{ccc}
(Y-A, X-A) & \xrightarrow{\quad i \quad} & (Y, X) \\
\Big\downarrow \text{proj.} & & \Big\downarrow \text{proj.} \\
(Y-A/X-A, \text{point} & \xrightarrow[\approx]{\quad j \quad} & (Y/X, \text{point})
\end{array}$$

The two projection maps induce isomorphisms for K^{-n}, so does j, because j is a homeomorphism. Thus $K^{-n}(i)$ is bijective.

In case that $A = X$, $Y/X = (Y-X)^{+}$. So $K^{-n}(Y-A, \emptyset) = K^{-n}(Y - X, \emptyset) = K^{-n}(Y, X)$.

Remark: The above excision axiom implies the one given by Eilenberg and Steenrod.

(D.10)
We introduce some operations in the category of compact spaces with base-point and base-point preserving maps. The base-point of a space is often denoted *. Let X, Y, Z be compact spaces with base-points x_0, y_0, z_0 respectively.

Definition: The join $X \vee Y$ of X and Y is a subspace of $X \times Y$:

$$X \vee Y = \{(x, y) \mid (x, y) \in X \times Y \text{ and } (x = x_0 \text{ or } y = y_0)\} .$$

The smash product $X \wedge Y$ of X and Y is a quotient space of $X \times Y$:

$$X \wedge Y = X \times Y/_{X \vee Y} .$$

Properties of $\vee$ and $\wedge$:

1. $X \vee Y$ and $X \wedge Y$ are compact spaces with base-point.
2. $\wedge$ and $\vee$ are commutative and associative.
3. $\wedge$ is distributive over $\vee$, i.e. $X \wedge (Y \vee Z) = (X \vee Y) \wedge (X \vee Z)$.

Only the following two properties are non-trivial:

The smash product is associative for compact spaces.

Proof: Let (X, x_0), (Y, y_0), (Z, z_0) be three spaces with base-points. We define $X \wedge Y \wedge Z$ (without brackets) as the product space $X \times Y \times Z$ with all points (x, y, z) with either $x = x_0$, $y = y_0$ or $z = z_0$ identified to one point, the base-point of $X \wedge Y \wedge Z$. We consider the following sequence of maps

$$X \times Y \times Z \to (X \times Y) \times Z \to (X \times Y/X \vee \ ) \times Z \to (X \times Y_{/X \vee Y}) \times Z /(X \times Y/X \vee Z) \vee Z$$
$$\| $$
$$(X \wedge Y) \wedge Z.$$

Then a point with either $x = x_0$, $y = y_0$ or $z = z_0$ will be mapped into the base-point of $(X \wedge Y) \wedge Z$. Therefore we have a continuous surjective map $X \wedge Y \wedge Z \to (X \wedge Y) \wedge Z$. This map is bijective, for if the point (x, y, z) is mapped into the base-point of $(X \wedge Y) \wedge Z$, then either $(x, y) = * \in X \wedge Y$ or $z = z_0$, whence $x = x_0$, $y = y_0$ or $z = z_0$. The space $X \wedge Y \wedge Z$ and $(X \wedge Y) \wedge Z$ are compact (because A/B is compact for any compact pair (A, B)), so they are homeomorphic. q.e.d.

The smash product $\wedge$ is distributive over the join $\vee$ for compact spaces.

Proof: Let (X, x_0) (Y, y_0), (Z, z_0) be compact spaces with base-point. Consider the composite map

$$X \times (Y \vee Z) \xrightarrow{d \times \infty} (X \times X) \times (Y \times Z) \to (X \times Y) \times (X \times Z) \to (X \wedge Y) \times (X \wedge Z).$$

(d is the diagonal map.) This is a continuous map. The image of this map is contained in $(X \wedge Y) \vee (X \wedge Z)$, for if in $(x, y, z) \in X \times (Y \vee Z)$ $y = y_0$, then $(x, y_0) = * \in X \wedge Y$ and if $z = z_0$, then $(x, z_0) = * \in X \wedge Z$. Therefore we have a continuous map $f : X \times (Y \vee Z) \longrightarrow (X \wedge Y) \vee (X \wedge Z)$. This map is surjective; e.g. let $(*, (x, z)) \in (X \wedge Y) \vee (X \wedge Z)$, then $f(x, y_0, z) = (*, (x, z))$. Now let $(x, y, z) \in X \times (Y \vee Z)$ and suppose that $f(x, y, z) = * \in (X \wedge Y) \vee (X \wedge Z)$, then $(x, y) = * \in X \wedge Y$ and $(x, z) = * \in X \wedge Z$, hence $x = x_0$ or, if $x \neq x_0$, $y = y_0$ and $z = z_0$. So we have:

$$f(x, y, z) = * \iff (x = x_0) \text{ or } ((y_0, z_0) = (y, z)) .$$

That means that f factorises through $X \wedge (Y \vee Z)$ such that the induced continuous map $\tilde{f} : X \wedge (Y \vee Z) \to (X \wedge Y) \vee (X \wedge Z)$ is injective. So we have a bijective continuous map $X \wedge (Y \vee Z) \to (X \wedge Y) \vee (X \wedge Z)$. Both the spaces are compact, so they are homeomorphic. q.e.d.

Lemma 1: $S^n \wedge S^m = S^{n+m}$.

Proof: Clear, because both $S^n \wedge S^m$ and S^{m+n} are the one-point-compactification of $\mathbb{R}^{n+m}$.

Definition: We define the "reduced suspension" $\tilde{\Sigma}X$ of a space X as $\Sigma X/\{x_0\} \times I$.

Lemma 2: $\tilde{\Sigma}X = S^1 \wedge X$

The proof is easy.

Corollary: If $p : \Sigma X \to S^1 \wedge X$ is the canonical projection, then $K'(\Sigma^n p)$ is bijective ($n = 0,1,2,\ldots$). The same is true, if p is the canonical map $p : \Sigma^m X \to S^m \wedge X$.

For a space X with base-point we define the point $[X \times \{0\}]$ as the base-point of ΣX. Consider the following commutative diagram:

$$\begin{array}{ccc} \Sigma X \vee \Sigma Y & \xrightarrow{\quad p \quad} & \Sigma(X \vee Y) \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle r} & \\ & \tilde{\Sigma}(X \vee Y) & \end{array}$$

p, q, r are canonical projections. By Lemma D.7, $K'(\Sigma^n q)$ and $K'(\Sigma^n r)$ are bijective. Hence $K'(\Sigma^n p)$ is an isomorphism. By induction we have the result:

Lemma 3: There is a canonical projection

$$\pi : \Sigma^n X \vee \Sigma^n Y \longrightarrow \Sigma^n(X \vee Y),$$

and $K'(\pi)$ is an isomorphism.

Note that the diagram

$$\begin{array}{ccc} \Sigma^n X \vee \Sigma^n Y & \xrightarrow{\quad \pi \quad} & \Sigma^n(X \vee Y) \\ & {\scriptstyle incl.}\nwarrow \quad \nearrow{\scriptstyle \Sigma^n i_1} & \\ & \Sigma^n X & \end{array}$$

where $i_1 : X \to X \vee Y$ is the inclusion, is commutative.

Lemma 4: If $i_1 : X \to X \vee Y$, $i_2 : Y \to X \vee Y$ are the inclusion maps, then $j : K'(X \vee Y) \to K'(X) \oplus K'(Y)$, defined by $j(\alpha) = (i_1^*\alpha, i_2^*\alpha)$, is an isomorphism.

Proof: Let $\alpha \in K'(X)$, $\beta \in K'(Y)$. Represent α, β by vector bundles E, F over X, Y respectively, which have the same dimension over the base-points. That is possible because of B.5 and the definition of K'. By clutching E and F together over the base-point of $X \vee Y$, we get a vector bundle G over $X \vee Y$. Put $k(\alpha,\beta) = [G]$; then

$k : K'(X) \otimes K'(Y) \longrightarrow K'(X \vee Y)$ is well-defined and the inverse of j.

Proposition: Let $p : X \times Y \longrightarrow X \wedge Y$ be the projection, $i : X \vee Y \longrightarrow X \times Y$ the inclusion. Then the following sequence is exact and splits for $n = 0,1,2,\ldots$:

$$0 \longrightarrow K'(\Sigma^n(X \wedge Y)) \xrightarrow{(\Sigma^n p)^*} K'(\Sigma^n(X \times Y)) \xrightarrow{(\Sigma^n i)^*} K'(\Sigma^n(X \vee Y)) \longrightarrow 0$$

Proof: Consider the exact sequence of D.8 for the pair $(X \times Y, X \vee Y)$. In view of that sequence it is sufficient to construct a homomorphism
$h_n : K'(\Sigma^n(X \vee Y)) \longrightarrow K'(\Sigma^n(X \times Y))$ for $n = 0,1,2,\ldots$ such that $(\Sigma^n i)^* \; h_n = \mathrm{id}$.
h_n is defined as the composition of the homomorphisms
$K'(\Sigma^n(X \vee Y)) \xrightarrow{\pi^*} K'(\Sigma^n X \vee \Sigma^n Y) \xrightarrow{j} K'(\Sigma^n X) \otimes K'(\Sigma^n Y) \xrightarrow{g} K'(\Sigma^n(X \times Y))$,
where π^* and j are defined according to the last two lemmas, and $g(\alpha,\beta) = (\Sigma^n\pi_1)^*\alpha + (\Sigma^n\pi_2)^*\beta$. ($\pi_i$ is the i-th projection of $X \times Y$.) An easy calculation shows that $j\pi^*(\Sigma^n i)^* g = \mathrm{id}$. But π^* and j are isomorphisms. q.e.d.

(D. 11)
In B. 8 we defined $\tilde{K}$ for spaces with base-point. A base-point preserving map $f : X \to Y$ induces a homomorphism $\tilde{K}(f) : \tilde{K}(Y) \to \tilde{K}(X)$. Now $\tilde{K}$ is a functor on the category of compact spaces with base-point and base-point preserving maps. Clearly, the restriction of K' to this category is naturally equivalent to $\tilde{K}$.

Let X, Y be compact spaces with base-points, π_i the i-th projection of $X \times Y$, $p : X \times Y \to X \wedge Y$, $i : X \vee Y \to X \times Y$ the natural maps. Define $\phi : K(X) \otimes K(Y) \to K(X \times Y)$ by $\phi(\alpha \otimes \beta) = \pi_1^*\alpha . \pi_2^*\beta$. The image of $\phi| \tilde{K}(X) \otimes \tilde{K}(Y)$ is in $\tilde{K}(X \times Y)$ as is easily seen. So ϕ induces a homomorphism $\tilde{\phi} : \tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X\times Y)$

Assertion: $\tilde{K}(i) \circ \tilde{\phi} = 0$.

Because of D.10 Lemma 4, it is sufficient to prove $j \circ \tilde{K}(i) \circ \tilde{\phi} = 0$. This is clear, becuae $\tilde{K}(\pi_2 i i_1)$ and $\tilde{K}(\pi_1 i i_2)$ are zero.
By D.10 Proposition, $\tilde{\phi}$ factorises through $\tilde{K}(X \wedge Y)$.

We now specialise $Y = S^2$. By Bott's periodicity theorem C.10 we know: For any $x \in \tilde{K}(X \wedge S^2)$ there are x_1 and $x_2 \in K(X)$, both uniquely determined, such that $x = \pi_1^* x_1 + \pi_1^* x_2 . \pi_2^*([H] - 1)$. Here H denotes the Hopf bundle over S^2. $[H] - 1$ is a generator of $\tilde{K}(S^2)$.

Theorem: The homomorphism

$$\psi : \tilde{K}(X) \longrightarrow \tilde{K}(X \wedge S^2),$$

defined by $\psi(\xi) = \pi_1^* \xi \cdot \pi_2^*([H]-1)$, is an isomorphism.

Proof: For the sake of convenience we write

$$\tilde{K}(X \wedge S^2) \subset \tilde{K}(X \times S^2) \subset K(X \times S^2).$$

Let $X = \pi_1^* x_1 + \pi_1^* x_2 \cdot \pi_2^*([H] - 1) \in \tilde{K}(X \wedge S^2)$. Then $x_1 = 0$ because $x \in \tilde{K}(X \times S^2)$. $x_2 \in \tilde{K}(X)$ because $i^* x = 0$. So $x \in$ image (ψ), and ψ is surjective. ψ is a monomorphism because of the Bott periodicity. q.e.d.

Remark: ψ is natural with respect to continuous maps preserving the base-point. This is a consequence of the naturality of the homomorphism ϕ defined above.

Corollary: $K(X) \simeq K(X \wedge S^2)$

We are now able to determine K and $\tilde{K}$ for spheres. $\tilde{K}(S^0) \simeq \mathbb{Z}$, so by D.10 Lemma 1: $\tilde{K}(S^{2n}) \simeq \mathbb{Z}$. $\Sigma(S^0) = S^1$, and $GL(m, \mathbb{C})$ is pathwise connected, therefore by A.22 we have $\tilde{K}(S^1) = 0$. $\tilde{K}(S^{2n+1}) = 0$ follows. For each n holds $K(S^n) \simeq \tilde{K}(S^n) \oplus \mathbb{Z}$.

(D.12)

Using the theorem of D.11 and the corollary of Lemma 2 (D. 10), we have an isomorphism $\chi : K^{-n}(Y, X) \to K^{-n-2}(Y, X)$ for any pair (Y, X) of compact spaces. As ψ, the above isomorphism is compatible with maps. The following diagram is commutative:

$$\begin{array}{ccccccc}
K^{-n}(Y, X) & \longrightarrow & K^{-n}(Y) & \longrightarrow & K^{-n}(X) & \xrightarrow{\delta} & K^{-n+1}(Y, X) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \\
K^{-n-2}(Y, X) & \longrightarrow & K^{-n-2}(Y) & \longrightarrow & K^{-n-2}(X) & \xrightarrow{\delta} & K^{-n-1}(Y, X)
\end{array}$$

To see that for the rightmost square, remember that δ is induced by a map $h : C_f \longrightarrow \Sigma X$ (cf. D.5). Then commutativity follows from the naturality of χ.

Consequence: For any compact pair (Y, X) we have an exact hexagon

$$\begin{array}{ccccc}
 & & K^0(Y, X) & & \\
 & \nearrow & & \searrow & \\
K^{-1}(X) & & & & K^0(Y) \\
\uparrow & & & & \downarrow \\
K^{-1}(Y) & & & & K^0(X) \\
 & \nwarrow & & \swarrow & \\
 & & K^{-1}(Y, X) & &
\end{array}$$

which is natural with respect to maps.

E. λ-rings; the operations ψ^k

(E.1) Definition

A commutative ring A with unity is called a λ-ring if there are given maps

$$\lambda^i : A \longrightarrow A \qquad i = 0,1,2,\ldots$$

such that for all $x,y \in A$

$$\lambda^0(x) = 1$$

$$(1) \qquad \lambda^1(x) = x$$

$$\lambda^n(x+y) = \sum_{i+j=n} \lambda^i(x)\,\lambda^j(y) \qquad i,j \geqq 0 .$$

Let $1 + A[[t]]^+$ denote the multiplicative group of formal power series in t with coefficients in A, starting with 1.

We define $\lambda_t : A \longrightarrow 1 + A[[t]]^+$ by $\lambda_t(x) = \Sigma_0^\infty \lambda^i(x)t^i$. Then the requirements (1) on the λ^i imply that λ_t is a non-trivial homomorphism of the additive group A into $1 + A[[t]]^+$.

(E.2) Example

Let A be an arbitrary commutative ring with unity. We define a canonical λ-ring structure on $\Lambda = 1 + A[[t]]^+$.

Let $x_1,\ldots,x_n; y_1,\ldots,y_n$ be a set of 2n variables, let σ^k denote the k^{th} elementary symmetric function.

Consider $\sigma^k(x_1y_1, x_1y_2,\ldots, x_1y_n, x_2y_1,\ldots, x_2y_n,\ldots, x_ny_1,\ldots,x_ny_n)$

this function is symmetric in $x_1,\ldots,x_n$ and in $y_1,\ldots,y_n$ and therefore can be written as a polynomial $u^k(\sigma^1(x_1,\ldots,x_n),\ldots, \sigma^k(x_1,\ldots,x_n),$ $\sigma^1(y_1,\ldots,y_n),\ldots, \sigma^k(y_1,\ldots, y_n))$ or simply $u^k(\sigma_x^1,\ldots, \sigma_x^k, \sigma_y^1,\ldots,\sigma_y^k)$.

The polynomial u^k is independent of n, provided $n \geqq k$.

We consider also $\sigma^m(\alpha_1,\ldots,\alpha_i)$ where α_j runs through all products $x_{i_1}x_{i_2}\ldots x_{i_r}$ with $i_1 < i_2 < \ldots < i_r$ this is again a symmetric function of the $x_1,\ldots,x_n$ and therefore can be written as a polynomial

$$v_r^m(\sigma_x^1,\ldots,\sigma_x^k) \qquad k = m.r$$

These polynomials are also independent of the number n of variables provided it is large enough.

We now define a multiplication in Λ, denoted 0, as follows

$$(1+ \sum_{i\geqq 1} a_i t^i) \circ (1 + \sum_{i\geqq 1} b_i t^i) = 1 + \sum_{k\geqq 1} u^k(a_1,\ldots,a_k, b_1,\ldots,b_k)t^k$$

The associativety of this operation is proved as follows.
We observe that if we have two polynomials of the form $\prod_{i=1}^{n}(1+\alpha_i t)$, $\prod_{i=1}^{n}(1+\beta_i t)$ and define a composition o by

$$\prod_{i=1}^{n}(1+\alpha_i t) \circ \prod_{i=1}^{n}(1+\beta_i t) = \prod_{i,j}(1+\alpha_i\beta_j t)$$

then the coefficient of t^k in the result is exactly

$$u^k(\sigma_\alpha^1,\ldots,\sigma_\alpha^k,\sigma_\beta^1,\ldots,\sigma_\beta^k).$$

Now this composition is visibly associative. This gives certain formula's for the u^k which prove the associativity of the composition in A.
The distributivity with respect to the addition in $\tilde{A}$ (= the usual multiplication of power series) is proved in like manner. 1+t = Neutrales El. bzgl. o
We remark that if we have two finite polynomials in $\tilde{A}$ $\tilde{a}= 1+\sum_{i=1}^{n} a_i t^i$ $\tilde{b}= 1+\sum_{i=1}^{m} b_i t^i$ then if we formally write $\tilde{a} = \prod_{i=1}^{n}(1+\alpha_i t)$, $\tilde{b}= \prod_{i=1}^{m}(1+\beta_i t)$ then

$$\tilde{\alpha}\circ\tilde{b} = \prod_{i,j}(1+\alpha_i\beta_j t) \quad .$$

We define a λ-ring structure on $\tilde{A}$ by setting

$$\lambda^r(1+\sum_{i\geq 1} a_i t^i) = 1+\sum_{k\geq 1} v_r^k(a_1,\ldots,a_m)t^k \qquad m= k.r$$

if we write an element $\tilde{a} = 1+\sum_{i=1}^{n} a_i t^i = \prod_{i=1}^{n}(1+\alpha_i t)$ then $\lambda^r(\tilde{a}) =$

$= \prod_{i_1<\ldots<i_r}(1+\alpha_{i_1}\ldots\alpha_{i_r} t)$. We prove now by the same kind of trick as for the associativity of o that the λ^r so defined satisfy the conditions (1) of (E.1)

(E.3) A λ-ring A is called a special λ-ring if the map $\lambda_t : A \to \tilde{A}$ is a homomorphism of λ-rings; that is λ_t is a ringhomomorphism, and if we denote the λ-operations in $\tilde{A}$ by capital Λ's then $\lambda_t(\lambda^i(a)) = \Lambda^i(\lambda_t(a))$ must be true for all $a \in A$. We try to find the conditions on the λ^i which make A into a special λ-ring:

$$\lambda_t(a.b) = (1+\sum_{i\geq 1}\lambda^i(a)t^i)\circ(1+\sum_{i\geq 1}\lambda^i(b)t^i) =$$

$$= 1+\sum_{k\geq 1} u^k(\lambda^1(a),\ldots,\lambda^k(a),\lambda^1(b),\ldots,\lambda^k(b))t^k$$

so neccessary and sufficient conditions to make λ_t a ring homomorphism are

$$\lambda^k(a,b) = u^k(\lambda^1(a),\ldots,\lambda^k(a),\lambda^1(b),\ldots,\lambda^k(b)) \qquad k\geq 1.$$

In the same way one finds that the commuting of λ_t with the λ-operations is equivalent with

$$\lambda^r(\lambda^i(a)) = v_i^r(\lambda^1(a),\ldots,\lambda^k(a)) \qquad k = r.i.$$

Proposition: If A is an arbitrary commutative ring with unity, then the λ -ring $\tilde{\Lambda}$ constructed in (E.2) is a special λ-ring.

<u>Proof</u>: If $\tilde{a} = \prod_{i=1}^{m} (1+\alpha_i t)$, $\tilde{b} = \prod_{i=1}^{n} (1+\beta_i t)$, then

$\lambda^r(\tilde{a} \circ \tilde{b}) = \Pi(1+\alpha_{i_1}\beta_{j_1}\alpha_{i_2}\beta_{j_2}\ldots\alpha_{i_r}\beta_{j_r}t)$ and $\lambda^1(a) = \sigma^1(1+\alpha_1 t,\ldots, 1+\alpha_n t)$

$\ldots \lambda^r (a) = \sigma^r(1 + \alpha_1 t,\ldots, 1+\alpha_n t)$. Now $\lambda^r(\tilde{a} \circ \tilde{b}) = \sigma^r(1 + \alpha_1\beta_1 t,\ldots, 1+\alpha_n\beta_m t)$

so that the relation asked for is indeed satisfied.

The proof is completed in the same way as in (E.2) The proof that $\lambda^r(\lambda^i(\tilde{a})) = v_i^r(\lambda^1(\tilde{a}),\ldots, \lambda^k(\tilde{a}))$ is left to the reader as an easy exercise.

(E.4) The operations ψ^k

Let A be any λ-ring. We introduce operations

$\psi^k : A \to A$ by setting

(1) $\psi_t(x) = \Sigma_1^\infty \psi^k(x)t^k$

and

(2) $\psi_{-t} = -t \frac{d}{dt} \ln \lambda_t = -t \frac{d}{dt}(\lambda_t).(\lambda_t)^{-1}$

(formal operations on power series.)

Then $\psi_{-t}(x+y) = -t\ \lambda'_t (x+y)/\lambda_t(x+y) =$

$= -t\{\lambda'_t (x)\lambda_t(y) + \lambda'_t (y)\lambda_t(x)\} / \lambda_t(x).\lambda_t(y) = \psi_{-t}(x) + \psi_{-t}(y)$

from which follows that the ψ^k are additive. If it happens that $\lambda_t(x) = 1+xt$ (i.e $\lambda^i(x) = 0 \quad i\geq 2$) then $\psi_t(x) = \frac{xt}{1-xt}$, and $\psi^k(x) = x^k$.

If s_k is the polynomial such that

$s_k(\sigma^1(\alpha_1,\ldots,\alpha_m),\ldots, \sigma^k(\alpha_1,\ldots,\alpha_m)) \equiv \alpha_1^k +\ldots+\alpha_m^k$, where σ^i denotes the i^{th} elementary symmetric functions (m sufficiently large), then

$$\psi^k(x) = s_k(\lambda^1(x),\ldots, \lambda^k(x))$$

for if we equate in the relation (2)

$$\psi_{-t}\lambda_t + t \frac{d}{dt} \lambda_t = 0$$

the coefficients of the diverse powers of t we get

$$\psi^1 - \lambda^1 = 0$$

$$\psi^2 - \psi^1\lambda^1 + 2\lambda^2 = 0$$

$$\psi^3 - \psi^2\lambda^1 + \psi^1\lambda^2 - 3\lambda^3 = 0$$

$$\cdots\cdots\cdots$$

$$\psi^n - \psi^{n-1}\lambda^1 + \dots + n\lambda^n = 0$$

and these are precisely the formulae linking the elementary symmetric functions, with the power sums.

(E.5) Lemma

Let A be a special λ-ring, then the ψ^k are ringhomomorphism, they commute with the λ^i and $\psi^k\psi^l = \psi^{kl} = \psi^l\psi^k$

<u>Proof:</u> In the ring $\tilde{A}$ the Λ^r and ψ^k are given by

$$\Lambda^r\left(\prod_{i=1}^{n}(1+\alpha_i t)\right) = \prod_{i<\dots<i_r}(1+\alpha_{i_1},\dots,\alpha_{i_r} t)$$

$$\psi^k\left(\prod_{i=1}^{n}(1+\alpha_i t)\right) = \prod_{i=1}^{n}(1+\alpha_i^k t)$$

and therefore the lemma is trivially true in the ring $\tilde{\lambda}$. Now $\lambda_t : A \to \tilde{\lambda}$ is an injective ringhomomorphism which commutes with the λ-operations. From the definitic of ψ^k as $\psi^k(x) = s_k(\lambda^1(x),\dots,\lambda^k(x))$ it follows that λ_t also commutes with the ψ^k, and the lemma is proven. q.e.d.

Remark: ψ^1 is the identity

(E.6) Definitions

Let A be a λ-ring, an element $x \in A$ is called of λ-dimension k iff $\lambda_t(x)$ is a polynomial of degree $k < \infty$ (i.e. if $\lambda^k(x) \neq 0$, $\lambda^l(x) = 0$ $l > k$).

We define also

$$\gamma_t = \lambda_{t/1-t} \qquad \text{(operations on formal series)}$$

$$\gamma_t(x) = \Sigma_0^\infty \gamma^i(x)t^i \qquad \gamma^0(x) = 1 \quad \gamma^1(x) = x$$

An immediate consequence is that

$$\gamma_t(x+y) = \gamma_t(x).\gamma_t(y) \quad .$$

An element $x \in A$ is called of γ-dimension k iff $\gamma_t(x)$ is a polynomial of degree k

Exercise: Let A be a special λ-ring, prove

$$\lambda_t(1) = 1+t, \quad \gamma_t(1) = \frac{1}{1-t} \quad \lambda_t(k) = (1+t)^k$$

$$\gamma_t(k) = (1-t)^{-k}, \; \gamma_t(-k) = (1-t)^k$$

Lemma: In a special λ-ring:

x has λ-dimension $\leq k$ iff $x-k$ has γ-dimension $\leq k$.

Proof: This follows from the identities

$$\lambda_t(x) = \lambda_t(x-k+k) = \lambda_t(x-k). \lambda_t(k) = \gamma_{\frac{t}{1+t}}(x-k)(1+t)^k$$

$$\gamma_t(x-k) = \gamma_t(x)\{\gamma_t(k)\}^{-1} = \lambda_{\frac{t}{1-t}}(x)(1-t)^k$$

For, if $\gamma^n(x-k) = 0$, for $n>k$, then $\gamma_{\frac{t}{1+t}}(x-k)(1+t)^k =$

$= \{\gamma^0(x-k) + \gamma^1(x-k)\frac{t}{1+t} + \ldots + \gamma^k(x-k)(\frac{t}{1+t})^k\}(1+t)^k$ = polynomial of degree $\leq k$ and vice versa q.e.d.

For any λ-ring

$$\gamma_t = \Sigma\lambda^i(\frac{t}{1-t})^i = \Sigma\lambda^i t^i(1-t)^{-i} = \Sigma\lambda^i t^i \sum_{n=0}^{\infty} \binom{-i}{n}(-t)^n$$

$$= \Sigma\lambda^i t^{n+i} \frac{i(i+1)\ldots(i+n+1)}{1.2.} = \Sigma\lambda^i t^{n+i} \binom{n+i+1}{n} =$$

$$\Sigma\lambda^i t^{n+1}\binom{n}{i}$$

So $\gamma^{n+1} = \Sigma\binom{n}{i}\lambda^i$

Exercise. If $\dim_\lambda(x)< \infty$ then $\dim_\lambda(x+1) = \dim_\lambda(x) + 1$
If $\dim_\gamma(x)< \infty$ then $\dim_\gamma(x-1) = \dim_\gamma(x) + 1$

If $\dim_\lambda(x+n) = m$ then $\dim_\gamma(x+n-m)<\infty$,
$\dim_\gamma(x+n-m+1) = \infty$ $\dim_\gamma(x+n-m-1) = \dim_\gamma(x+n-m) + 1$.

In that case $x_0 = x+n-m$ is called the reduction of x and $\dim_\gamma(x+n-m) \overset{Df}{=} \dim_{\gamma_0}(x)$ $= \dim_\gamma(x_0)$ is called the reduced γ-dimension of x.

If $\dim_\lambda(\xi) = 1$, then ξ is called a <u>linebundle</u>

Then $\lambda_t(\xi) = 1 + \xi t$, $\gamma_t(\xi-1) = 1+(\xi-1)t$, $\psi^k(\xi) = \xi^k$.

If A is a special λ-ring, then $1 \in A$ is a linebundle.

(E.7) <u>The splitting principle for special λ-rings.</u> Let A be a special λ-ring, let $x \in A$ be of finite λ-dimension $\lambda_t(x) = 1+a_1t+\ldots+a_nt^n$. Consider the ring $\overline{A} = A[\alpha_1,\ldots,\alpha_n]/(\sigma^1(\alpha_1,\ldots,\alpha_n) - a_1,\ldots, \sigma^n(\alpha_1,\ldots,\alpha_n)-a_n)$.

Then we have an injection $A \to \overline{A}$ and in $\overline{A}[t]$ $1 + a_1t+\ldots+a_nt^n = \prod_{i=1}^{n}(1+\alpha_it)$.

Now $\tilde{\Lambda} \hookrightarrow \tilde{\bar{A}}$. $\tilde{\bar{A}}$ is a special λ-ring.

A is a special λ-ring, so $\lambda_t : A \to \tilde{\Lambda}$ is a ring homomorphism.

So we have embedded A into a special λ-ring $\tilde{\bar{A}}$ in such a way that x, considered as an element of $\tilde{\bar{A}}$, is a sum of linebundles.

(E.8) Proposition (Adams)

Let A be a special λ-ring, $\rho \in A$ of finite γ-dimension (i.e $\gamma^i(\rho) = 0$ $i > n$, some n) then the element

$$(\psi^k - k^d)(\psi^k - k^{d-1})\ldots(\psi^k - k)(\psi^k - 1)\rho$$

can be written as a linear combination of monomials $\gamma^{i_1}(\rho)\ldots\gamma^{i_r}(\rho)$ with $i_1+\ldots+i_r$

Proof: We check the result for a linebundle ξ by induction on d. We put $\xi = x+1$, b we use expressions in $\xi-1$ instead of x, x is the reduction of ξ. A is a special λ-ring so the ψ^k are ringhomomorphisms and $\lambda_t(1) = 1+t$, and therefore $\psi^k(1) = 1$, $(\psi^k - 1)(1) = 0$, so we will start from the <u>inductive hypothesis</u> (with ξ instead of $\xi-1$ in the left hand side).

$$(\psi^k - k^{d-1})\ldots(\psi^k - k)(\psi^k - 1)\xi = (\xi-1)^d\, p(\xi)$$

where p is a polynomial, and where we observe that

$$\gamma\,(\xi-1) = \xi-1, \quad \gamma^i(\xi-1) = 0 \qquad i > 1 \quad .$$

Then we have

$$(\psi^k - k^d)\ \ldots\ (\psi^k - 1)(\xi) = (\psi^k - k^d)[(\xi-1)^d\, p(\xi)] =$$

$$(\xi^k - 1)^d\, p(\xi^k) - k^d(\xi-1)^d\, p(\xi) = (\xi-1)^d q(\xi) \quad .$$

Now $q(\xi) = (\Sigma_0^{k-1}\xi^r)^d\, p(\xi^k) - k^d p(\xi)$ has a factor $\xi-1$ since $q(1) = (1+\ldots+1)^d p(1) - k^d p(1) = 0$.

The initial case d=0 yields since $\lambda_t(\xi) = 1+\xi t$ and therefore $\psi^k(\xi) = \xi^k$

$$(\psi^k - 1)(\xi) = \xi^k - \xi = (\xi-1)(\xi^{k-1}+\ldots+\xi) = (\xi-1)\ p(\xi)$$

and therefore is true.

This completes the inductive proof.

Next we check the result for a sum of linebundles

$$\rho = \xi_1+\ldots+\xi_n \quad .$$

If we set $\xi_r = 1+x_r$ as above, we have

$$(\psi^k - k^d)\ \ldots\ (\psi^k - 1)\rho = \sum_{r=1}^{n} (x_r)^{d+1}\, p(x_r)$$

where p is a suitable polynomial, the expression $\sum_{r=1}^{n} (x_r)^{d+1}\, p(x_r)$ is a symmetric polynomial in the x_r's whose homogeneous components have degree $>d$, by what we have just proved. Therefore when we write it as a linear combination of monomials

$\gamma^{i_1}(\rho)\dots\gamma^{i_1}(\rho)$, $i_1+\dots+i_r > d$ will hold.

That it is sufficient to consider finite sums of linebundles follows from the splitting principle of (E.7) .

Let $x \in A$ be of finite λ-dimension, $A \to \tilde{A}$ such that in $\tilde{A}$ x is a sum of line bundles, $\tilde{A}$ is a special λ-ring. So we have the diagram of rings and ringhomomorphisms

$$\begin{array}{ccc} A & \xrightarrow{\lambda_t} & \tilde{A} \\ \downarrow{\scriptstyle \psi^k} & & \downarrow{\scriptstyle \psi^k} \\ A & \xrightarrow{\lambda_t} & \tilde{A} \end{array}$$

The element $x \in A$, considered as an element of $\tilde{A}$, is a sum of linebundles. q.e.d.

(E.9) Definition

The special λ-ring A is said to be of reduced γ-dimension $\leq d$ if for every ρ of finite γ-dimension the monomial $\gamma^{i}(\rho),\dots,\gamma^{i_r}(\rho) = 0$ if $i_1+\dots+i_r > d$.

We then have the following corollary of (E.8)

In a special λ-ring A of reduced γ-dimension $\leq d$

$$(\psi^k - k^d)(\psi^k - k^{d-1})\dots(\psi^k - 1) = 0 \text{ .}$$

(E.10) Theorem (Adams)

If the special λ-ring A has reduced γ-dimension $\leq d$, then $A \otimes \mathbb{Q}$ (which is a vector space over $\mathbb{Q}$) splits as a direct sum $A \otimes \mathbb{Q} = \bigoplus_{q=0}^{d} V_q$ where $\psi^k(r) = k^q . r$ for $r \in V_q$ all $k > 0$.

<u>Proof</u>: Fix a prime k. By (E.9) and linear algebra $A \otimes \mathbb{Q}$ splits as a direct sum of eigenspaces $V^{(k)}_q$ belonging to ψ^k. Let $l \neq k$ be such that l is not divisible by k; ψ^k and ψ^l commute so $V^{(k)}_q$ splits into eigenspaces V^{kl}_{qr} corresponding to the eigenvalues l^r of ψ^l. On V^{kl}_{qr}, $\psi^l\psi^k$ is a multiplication by $k^q l^r$. But we know from (E.5) that $\psi^k\psi^l = \psi^{kl}$, which means that $\psi^k\psi^l$ has only the eigenvalues $(kl)^s$ Therefore only r=q contributes essentially in $V^{kl}_{q\,r}$ *) and $V^{kl}_{qq} = V^k_q = V^l_q$. If we take any two natural numbers l,l' there is always a prime k such that neither l nor l' is divisible by k. Therefore V^k_q does not depend on k, and we can call $V^k_q = V_q$ to get the theorem. q.e.d.

*) that is $V^{kl}_{qr} = 0$ if $q \neq r$.

(E.11) λ-rings with augmentation

The definition $\lambda_t(1) = 1+t$ givens $\mathbb{Z}$ (the ring of the integers) the structure of a special λ-ring: $\lambda_t(k) = (1+t)^k$, $\lambda^r(n) = \binom{n}{r}$.

A λ-ring A together with a homomorphism of λ-rings $\varepsilon : A \to \mathbb{Z}$ is called a <u>λ -ring with augmentation</u>.

Grothendieck defines for every λ-ring A with augmentation a filtration in the following way. Let I be Ker ε , then A_n is the subgroup generated by the monomials

$$\gamma^{n_1}(x_1).\ \gamma^{n_2}(x_2),\ldots,\ \gamma^{n_k}(x_k)$$

with $x_i \in I$, $\sum_{i=1}^{k} n_i \geq n$.

(cf [2])

F) <u>Applications of the theory of λ-rings to K(X)</u>

(F.1) ring Proposition,

K(X) is a λ-ring with augmentation (X is connected)

Proof: define

$$\lambda_t : B(X) \longrightarrow 1 + K(X)[[t]]^+ \text{ by}$$
$$E \rightsquigarrow \sum_{0}^{\infty} [\lambda^i(E)]t^i$$

($[\lambda^i(E)]$ denotes the element in K(X) represented by $\lambda^i(E) \in B(X)$.

Then because of the identity (1) in (E.1) we have that

$$\lambda_t(E).\lambda_t(E') = \lambda_t(E + E') .$$

So λ_t is an additive homomorphism in the abelian group $1 + K(X)[[t]]^+$ and therefore factorizes through K(X). This defines the λ^i for all elements in K(X) and makes K(X) a λ-ring. The explicit formula for an element $\xi = \sum_{<\infty} n_i[E_i] \in K(X)$ is

$$\lambda_t(\xi) = \Pi \lambda_t[E_i]^{n_i}$$

We define $\varepsilon : B(X) \longrightarrow \mathbb{Z}$ by $\varepsilon(E) = \dim E$. ε is clearly an additive homomorphism and so factorizes through K(X). This defines the augmentation of K(X). q.e.d.

(F.2) The splitting principle

Let E be a complex vectorbundle over X, we define $\mathbb{P}(E)$ as follows. $\mathbb{P}(E)_x$ = {all lines in E_x through origin of E_x} i.e. $\mathbb{P}(E)_x$=the projective spaces associated to E_x.

$\mathbb{P}(E) = \mathbb{P}(E)_x$ with the local product topology inherited from E.

The map $\mathbb{P}(E) \xrightarrow{\pi} X$ defined by mapping each line l_x of E_x into x is a fibering of X.

There is a canonical linebundle over $\mathbb{P}(E)$.

S_E, the linebundle, whose fiber over $l_x \in \mathbb{P}(E)$ consists of the points of l_x.
And there is another bundle

Q_e, the bundle, whose fiber over l_x $\mathbb{P}(E)$ consists of the vectorspace $E_x/1$
We have now over $\mathbb{P}(E)$ an exact sequence of bundles

$$0 \longrightarrow S_E \longrightarrow \pi^{-1}E \longrightarrow Q_E \longrightarrow 0 .$$

So $[\pi^{-1}E] = [S_E] + [Q_E]$ in $K(\mathbb{P}(E))$ dim $Q_E = n-1$ if dim $E=n$.
Set $E_1 = Q_E$ and consider $\mathbb{P}(E_1)$ over $\mathbb{P}(E)$. When E is lifted to $\mathbb{P}(E_1)$ it splits off two linebundles. Continuing this process we find a space $\mathbb{P}(E_n)$ such that when E is lifted to $\mathbb{P}(E_n)$ it can be written as a sum of n linebundles.

The composed map $\rho : \mathbb{P}(E_n) \longrightarrow \mathbb{P}(E_{n-1}) \to \cdots \to \mathbb{P}(E) \longrightarrow X$
induces an injection $K(\rho)$ $K(X) \longrightarrow K(\mathbb{P}(E_n)) = K(\mathbb{F}(E))$.

This is known as the splitting principle. (cf. [8])

(F.3) Lemma

Let $f : X \to Y$ be a continuous map, then $K(f)$ $K(Y) \to K(X)$ is a homomorphism of λ-rings.
(i.e to prove $\lambda^i(f^{-1}E) = f^{-1}(\lambda^i E)$.

Proof: Exercise.

(F.4) K(X) is a special λ-ring.

First we prove that λ_t is a ringhomomorphim .

Let ξ_1, ξ_2 be two linebundles then $\xi_1 \otimes \xi_2$ is also a linebundle, and so
$\lambda_t(\xi_1 \otimes \xi_2) = 1 + \xi_1 \otimes \xi_2 t$. So $\lambda_t(\xi_1) \circ \lambda_t(\xi_2) = (1+\xi_1 t) \circ (1+\xi_2 t) =$

$= 1 + \xi_1\xi_2 t = \lambda_t(\xi_1 \otimes \xi_2)$.

Now consider the product of two sums of linebundles

$$(\xi_1 \oplus \ldots \oplus \xi_n) \otimes (\eta_1 \oplus \ldots \oplus \eta_m) = (\xi_1\eta_1 \oplus \xi_1\eta_2 \oplus \ldots \oplus \xi_1\eta_m \oplus \ldots \oplus \xi_n\eta_m)$$

λ_t being additive we get.

$$\lambda_t((\xi_1 \oplus \ldots \oplus \xi_n) \otimes (\eta_1 \oplus \ldots \oplus \eta_m)) = \prod_{i,j} (1+\xi_i\eta_j t) =$$

$$\Pi(1+\xi_i t) \circ \Pi(1+\eta_j t) = \lambda_t(\xi_1 \oplus \ldots \oplus \xi_n) \circ \lambda_t(\eta_1 \oplus \ldots \oplus \eta_m).$$

Now if we have two arbitrary bundles E,F over X we can find by the splitting principle for vectorbundles a space X_1 and a map $X_1 \to X$ such that the induced map

$K(X) \to K(X_1)$ is an injection, such a map is always a homomorphism of λ-rings, and such that E considered as an element of $K(X_1)$ splits in a sum of linebundles, considering F as an element of $K(X_1)$ we can repeat this procedure, to get a space X_2 and an injection

$$K(X) \to K(X_2)$$

such that both E, F split into a sum of linebundles.

If follows from this that

$\lambda_t(E \otimes F) = \lambda_t(E) \circ \lambda_t(F)$ for arbitrary bundles over X.

So the map λ_t: $B(X) \to 1 + K(X)[[t]]^+$

is a homomorphism of semirings, from which follows that the associated map

$\lambda_t : K(X) \longrightarrow 1+K(X)[[t]]^+$ is a ringhomomorphism.

Next we prove that λ_t is a homomorphism of λ-rings. Let $\xi = \xi_1 \oplus \xi_2 \ldots \oplus \xi_n$ be a sum of linebundles.

Then $\lambda^r(\xi_1 \oplus \xi_2 \ldots \oplus \xi_n) = 0$ if $r > n$

$\lambda_t(\xi_1 \oplus \xi_2 \oplus \ldots \oplus \xi_n) = \prod_{i=1}^{n} (1+\xi_i t)$, and $\lambda^r(\prod_{i=1}^{n} (1+\xi_i t)) = 0$ if $r > n$ by definition of λ^r in $1 + K(X)[[t]]^+$.

Let $r \leq n$, then

$$\lambda^r(\xi_1 \oplus \xi_2 \ldots \oplus \xi_n) = \sum_{i_1+i_2+\ldots+i_n=r} \lambda^{i_1}(\xi_1), \ldots, \lambda^{i_n}(\xi_n)$$

Now $\lambda^i(\xi_j) = 0$ if $i \geq 2$, so we have only to extend the sum over those $(i_1, \ldots, i_n)$ with $i_1+\ldots+i_n = r$ and $i_j = 0,1$ for every j. $\lambda^1(\xi_j) = \xi_j$, $\lambda^0(\xi_j) = 1$ for all j.

So $\lambda^r(\xi_1 \oplus \ldots \oplus \xi_n) = \sigma^r(\xi_1, \ldots, \xi_n)$

λ_t is a ringhomomorphism so $\lambda_t(\lambda^r(\xi)) = \sigma^r(\lambda_t(\xi_1), \ldots, \lambda_t(\xi_n))$

(The operation in the polynomial σ^r now being . and $\circ$)

$\lambda_t(\xi) = \prod_{i=1}^{n} (1+\xi_i t)$, so

$$\lambda^r(\prod_{i=1}^{n} (1+\xi_i t)) = \prod_{i_1<\ldots<i_r} (1+\xi_{i_1} \ldots \xi_{i_r} t) = \sigma^r(1+\xi_1 t, \ldots, 1+\xi_n t) =$$

$$= \sigma^r(\lambda_t(\xi_1), \ldots, \lambda_t(\xi_n)).$$ So $\lambda^r \lambda_t(\xi) = \lambda_t \lambda^r(\xi)$.

Again by the splitting principle this proves that $\lambda^r \lambda_t(E) = \lambda_t \lambda^r(E)$ for an arbitrary bundle over X.

Now as is easily checked $\lambda^r(1-t+t^2-t^3+\ldots) = \begin{cases} 1+t & \text{if } r \text{ even} \\ (1-t+t^2\ldots, & \text{if } r \text{ is odd} \end{cases}$

So $\lambda^r \lambda_t(-1) = \lambda_t \lambda^r(-1)$. Every element in K(X) can be written in the form $[E]-n.1$

The formulas for λ^i of a sum are the same in both rings, λ_t is a ring homomorphism so $\lambda_t\lambda^r(\alpha) = \lambda^r\lambda_t(\alpha)$ for every $\alpha\in K(X)$ q.e.d.

(F.5)

Topological filtration of K(X). (cf. [2] §2).

Let X be a finite CW complex, we introduce a filtration on K(X) by putting

$$K_p(X) = \mathrm{Ker}(K(X) \longrightarrow K(X^{p-1}))$$

where X^{p-1} denotes the (p-1)-skeleton of X.

This filtration is a homotopy invariant and turns K(X) into a filtered ring, i.e. $K_p(X).K_q(X) \subset K_{p+q}(X)$.

(F.6) Proposition

(cf. Atiyah Characters and cohomology of finite groups).

Let X be a finite CW-complex, if we denote the filtration of K(X) as an augmented λ-ring by $K'_n(X)$ we have

$$K'_n(X) \subset K_{2n}(X) \quad \text{for all } n.$$

(F.7) Corollary

K(X) is of reduced γ-dimension $\leq [\frac{1}{2}\dim(X)]$

Proof: Let $x\in K(X)$ be such that $x+n$ is of λ-dimension n, this means that $\varepsilon(x) = 0$. For every element in K(X) can be written $[E]-m$, with E a vectorbundle, and for a vectorbundle F $\varepsilon(F) = n$ is equivalent to "F is of λ-dimenison n"

So if $i_1+\ldots+i_r \geq [\frac{1}{2}\dim(X)]+1 \Rightarrow \gamma^{i_1}(x),\ldots,\gamma^{i_r}(x)$ K'

$\subset K_{\dim(X)+1} = 0$. So if $i_1+\ldots+i_r > |\frac{1}{2}\dim(X)|$ then $\gamma^{i_1}(x),\ldots,\gamma^{i_1}(x) = 0$.

q.e.d.

(F.8) Corollary

$K(X)\otimes\mathbb{Q}$ (a finite dimensional vector space over $\mathbb{Q}$, since K(X) is finitely generated, X being a finite CW complex cf. F.9) splits as a sum of the rational cohomology groups. $H^{2i}_k(X,\mathbb{Q})$ $(H^{2i}_k(X,\mathbb{Q}) = \mathrm{Ker}(\psi^k - k^i)$ by definition).

Proof: apply (E.10)

(F. 9) Proposition.

Let X be a finite CW-complex with only even dimensional cells. Then

1. $K^{-1}(X) = 0$
2. K(X) is a finitely generated free abelian group with as many generators as there are cells in X.

Proof: By induction, let $K^{-1}(X^{2n-2}) = 0$ as an induction hypothesis. We have the exact sequence.

$$K^{-1}(X^{2n}, X^{2n-1}) \longrightarrow K^{-1}(X^{2n}) \longrightarrow K^{-1}(X^{2n-1}) = K^{-1}(X^{2n-2})$$

X^{2n}/X^{2n-1} is a bouquet of even-dim. spheres, so $K^{-1}(X^{2n}, X^{2n-1}) = 0$ (use D.10). $K^{-1}(X^{2n-2}) = 0$ by induction, so $K^{-1}(X^{2n}) = 0$. X being finite dimensional it follows that $K^{-1}(X) = 0$.

For 2 we consider the exact sequence

$$K^{-1}(X^{2n-2}) \longrightarrow K(X^{2n}, X^{2n-2}) \longrightarrow K(X^{2n}) \longrightarrow K(X^{2n-2}) \longrightarrow K^{-1}(X^{2n}, X^{2n-2}).$$

By 1, $K^{-1}(X^{2n-2}) = 0 = K^{-1}(X^{2n}, X^{2n-2})$ If we suppose by induction that $K(X^{2n-2})$ is free abelian with as many generators as there are cells in X^{+2n-2}, then $K(X^{2n-2})$ is projective and the sequence splits yielding

$$K(X^{2n}) \simeq K(X^{2n-2}) \oplus K(X^{2n}, X^{2n-2}) \qquad \text{q.e.d.}$$

(F. 10)

We now consider very special complexes

$$X = e_0 \cup e_{2n} \cup e_{4n}$$

(Examples are the projective planes)

$K(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$

and we have a filtration (cf. (F.5))

$$K(X) \supset K_{2n}(X) \supset K_{4n}(X) \supset 0$$

each terming being $\mathbb{Z}$ less than the preceding one.

We can choose an additive base of K(X) by taking $1 \in K(X)$, $a \in K_{2n}(X)$, $b \in K_{4n}(X)$.

We know the multiplicative structure if we know the μ of

$$a^2 = \mu b \qquad (ab = ba = b^2 = 0 \quad K(X)$$

being a filtered ring).

(F. 11) Lemma

$$K_{2m}(X) = \mathrm{Ker}([K(X) \to K(X^{2m-1})]) = \mathrm{Ker}([K(X) \to K(X^{2m-2})])$$
$$= \mathrm{Ker}[(\Psi^n - k^n)(\quad)\ldots(\Psi^n - k^m)] \qquad \text{if } \dim X = 2n$$

(F. 12) Lemma

Let $k = 2^p q$, q odd, then the number r of factors 2 in 3^k-1 is

$$r = 1 \quad \text{if } p = 0$$
$$r = p + 2 \quad \text{if } p \geq 1$$

Proof: Let $p=0$, then k is odd

$(3^k-1) = 2.(3^{k-1} + 3^{k-2}+\dots+3+1)$. The second term of the product is the sum of k odd numbers. k is odd, hence the second term is odd.

Let $p=1$, then

$(3^k-1) = 8\ (3^{k-2} + 3^{k-4}+\dots+3^2+1)$. The second term is the sum of $\frac{k}{2}$ odd numbers. $\frac{k}{2}$ is odd. Hence $(3^{k-2}+\dots+3^2+1)$ is odd and 3^k-1 has exactly 3 factors 2.

Suppose now the lemma proved for $p\leq n-1$. $n\geq 1$.

Then $(3^{2^n\cdot q}-1) = (3^{2^{n-1}q}-1)(3^{2^{n-1}q}+1)$. By the induction hypothesis $3^{2^{n-1}q}-1$ has exactly $n+1$ factors 2 $n+1>1$, therefore $3^{2^{n-1}q}+1$ has only one factor 2. So $3^{2^n\cdot q}-1$ has $n+2$ factors 2. q.e.d.

Corollary. If $r \geq k$, then $k = 1,2,4$.

If $p \geq 3$ then $2^p > p+2$. If $p = 0,1,2$ r.s. then we get $k = 1,2,4$ q.e.d.

(F.13)

Theorem

The only spaces of the type considered in (F.10) with $\mu = 1$ have $n=1,2,4$.

Proof: $\psi^2(a) = 2^n a + \lambda a^2$ since $(\psi^2 - 2^n)\in K_{4n}(X)$ by (F.11) in the same way

$\psi^3(a) = 3^n a + \mu . a^2$

$$\psi^2\psi^3(a) = \psi^2(3^n a + \mu a^2) = 3^n(2^n a +\lambda a^2) + \mu(2^{2n}.a^2) \qquad (a^3 = 0)$$

$$\psi^3\psi^2(a) = 2^n(3^n a + \mu a^2) + \lambda 3^{2n} a^2$$

$\psi^2\psi^3 = \psi^3\psi^2$, therefore $\lambda 3^n + \mu . 2^{2n} = 2^n\mu + 3^{2n}\lambda$ or

$$\lambda 3^n(3^n-1) = \mu 2^n(2^n-1)$$

Now $\psi^2 = (\lambda^1)^2 - 2\lambda^2$ $\qquad (\psi^2 = \sigma^2(\lambda^1, \lambda^2))$

so $\psi^2(x) = x^2 - 2\lambda^2(x)$, so modulo 2 ψ^2 is the square of x.

so λ must be odd.

Therefore (3^n-1) must be divisible by 2^n.

By (F.12) this means $n=1,2,4$

(F.14) $\gamma(f)$

Consider a map $f : S^{4n-1} \longrightarrow S^{2n}$. We define $\gamma(f)$ as the linking number of $f^{-1}(p)$, $f^{-1}(q)$, for two points $p,q \in S^{2n}$ (approximating f by a differentiable function). Hopf defined maps with $\gamma(f) = 1$ for $n=1,2,4$ (using complex numbers, quaternions, Cayley numbers resp.)

Problem: are there maps with $\gamma(f) = 1$ for other n.

Suppose there is a map $f : S^{4n-1} \longrightarrow S^{2n}$ consider the diagram

$$\begin{array}{ccc} e_0 \cup e_{2n} & & E_{4n} \\ & \nwarrow f & \cup \\ & & \dot{E}_{4n} = S^{4n-1} \end{array}$$

We attach E_{4n} to $e_0 \cup e_{2n}$ by means of f, to get a CW-complex of the type $e_0 \cup e_{2n} \cup e_{4n}$. It was proved by Steenrod that with this construction and the using the notation introduced in F 10 that

$$a^2 = \pm \gamma(f)\ b$$

Applying now (F.13) we see that the problem posed by Hopf is solved.

There are no maps $f : S^{4n-1} \longrightarrow S^{2n}$ with $\gamma(f) = 1$ for n not equal to 1,2,4.

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