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THE RIEMANN–ROCH THEOREM FOR ANALYTIC EMBEDDINGS

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INTRODUCTION

IN [7] GROTHENDIECK formulated and proved a generalization of the Riemann–Roch theorem which we shall refer to as GRR. This theorem is concerned with a proper morphism $f: Y \to X$ of algebraic manifolds (any ground field) and reduces to the version (HRR) given in [14] when X is a point (and the ground field is $\mathbf{C}$). It is not known whether GRR or even HRR holds for arbitrary complex manifolds†. However the proof of GRR given in [7] breaks up into two separate cases:

(i) f is an embedding,

(ii) f is a projection $X \times P_N \to X$, where P_N is a projective space,

and the main purpose of this paper is to give a proof of GRR in case (i) for arbitrary complex manifolds. This proof is quite different from, and in many ways simpler than, that of [7] and, for the complex algebraic case, it gives a new proof‡ of GRR.

In [12] Grothendieck gives a more precise version of GRR in case (i) for characteristic zero. This also will be extended to the complex analytic case, and in some ways our version will be still more precise‡ than that of [12].

Another purpose of this paper is to relate GRR in the complex analytic and algebraic cases to the "differentiable GRR" of [3]. Roughly speaking the position is as follows. Let X, Y be *compact* (projective) algebraic manifolds, $f: Y \to X$ a morphism. Then we have the algebraic Grothendieck group $K^a(X)$, defined in [7], and the "topological Grothendieck group" $K(X)$ defined in [3]. Grothendieck defines, by use of sheaves, a homomorphism:

$$f_!^a : K^a(Y) \to K^a(X)$$

and GRR asserts that the diagram

$$\begin{array}{ccc} K^a(Y) & \xrightarrow{f_!^a} & K^a(X) \\ {\scriptstyle \mathrm{ch}}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^*(Y;\mathbf{Q}) & \xrightarrow{f_*} & H^*(X;\mathbf{Q}) \end{array} \qquad (1^a)$$

† Using a recent result of Grauert [11], and some of the ideas in this paper, it is not difficult to show that GRR can at least be *formulated* in this case.

‡ See the Remark after (3.1) for a more detailed discussion.

is commutative up to multipliers $\mathfrak{T}(Y)$ and $\mathfrak{T}(X)$ on the two sides. In [3] we showed that one could define, by quite different methods, a homomorphism

$$f_! : K(Y) \to K(X)$$

giving rise to a diagram

$$\begin{array}{ccc} K(Y) & \xrightarrow{f_!} & K(X) \\ \downarrow \mathrm{ch} & & \downarrow \mathrm{ch} \\ H^*(Y; \mathbf{Q}) & \xrightarrow{f_*} & H^*(X; \mathbf{Q}) \end{array} \tag{I}$$

In a more detailed publication we shall show that $f_!$ can be defined functorially and has similar formal properties to $f_!^a$. Granted this we shall show, in this paper, that diagram (I^a) is the composition of a commutative diagram

$$\begin{array}{ccc} K^a(Y) & \xrightarrow{f_!^a} & K^a(X) \\ \downarrow & & \downarrow \\ K(Y) & \xrightarrow{f_!} & K(X) \end{array}$$

and diagram (I).

Besides the intrinsic interest of GRR there are some interesting applications of GRR, in case (i), in recent work of Porteous [17, 18]. In [17] Porteous considers the following problem. Let X be an algebraic manifold, Y a submanifold and let X' be obtained from X by "blowing up" Y; the problem is to express the Chern classes of X' in terms of those of X and Y. A formula had been conjectured by previous authors and Porteous establishes the validity of this formula by use of GRR, for the injection $Y' \to X'$, where Y' is the counter-image of Y. Once we have extended GRR to deal with arbitrary compact complex manifolds the proof of Porteous goes over without change to the case when X is any compact complex manifold. Moreover there is one respect in which the proof of Porteous is incomplete and our results are needed to fill the gap.†

Because of these applications we also give a brief treatment of the real-analytic and C^∞ cases. These allow the results of Porteous to be extended to real-analytic and C^∞-manifolds.

The lay-out of the paper is as follows. In §1 we compare algebraic, holomorphic and real-analytic resolutions of coherent sheaves, and prove the compatibility of the correspon- ding "Grothendieck elements". In §2 we establish a basic lemma concerning the neighbour- hood of a complex submanifold of a complex manifold. The main theorems for embeddings then follow in §3. In §4 we compare $f_!$ and $f_!^a$ and prove the results outlined above. In §5 and §6 we deal briefly with real bundles and differentiable sheaves.

This paper is very closely related to [5], and we shall make free use of the notation and results of [5]. In fact this paper and [5] both rest on the same basic construction—that of the Grothendieck element [5; §4]—but whereas here we are concerned with *submanifolds*, in [5] the interest was essentially in subspaces with *singularities*.

† See §3 for a more detailed discussion.

§1. COMPATIBILITY OF GROTHENDIECK ELEMENTS

In [5] we introduced, for any pair of spaces X, Y with $Y \subset X$, the group $\mathscr{K}(X, Y)$; this extended the definition of the group $K(X, Y)$ given in [3] for a pair of finite CW-complexes. Roughly speaking $\mathscr{K}$ can be described as an "inverse limit singular theory". An analogous cohomology theory $\mathscr{H}$ was also defined in [5]. The following properties of $\mathscr{H}$, $\mathscr{K}$ are easily verified from the definitions:

(1.1) *If Y is a differentiable submanifold of a differentiable manifold X, then we have a Gysin homomorphism*

$$i_* : \mathscr{H}^*(Y) \to \mathscr{H}^*(X, X - Y)$$

with the usual properties.

(1.2) *$\mathscr{H}$ and $\mathscr{K}$ satisfy the usual excision isomorphism.*

Let X be a real-analytic manifold†, $\mathscr{O}$ the sheaf of germs of complex-valued real-analytic functions on X. If S is a coherent sheaf of $\mathscr{O}$-modules with support in Y, then we showed in [5; §4] that one could associate to S a definite "Grothendieck element" $\gamma_Y(S) \in \mathscr{K}(X, X - Y)$. If in particular we take $Y = X$, then we shall write $\gamma(S)$ for $\gamma_X(S) \in \mathscr{K}(X, \varnothing) = \mathscr{K}(X)$.

The definition of $\gamma_Y(S)$ used resolutions of S by locally free sheaves and the generalized difference bundle. From [5; (3.4)(viii)] we then get

PROPOSITION (1.3). *Let S be a coherent sheaf of $\mathscr{O}$-modules on X, with support in Y and let E be a real-analytic complex vector bundle on X. Then*

$$\gamma_Y(S) . E = \gamma_Y(S \otimes L),$$

where L is the locally free sheaf associated to E.

If X is a complex manifold then one can consider sheaves of $\mathscr{B}$-modules, where $\mathscr{B}$ is the sheaf of germs of holomorphic functions. If X is an algebraic manifold one can consider sheaves of $\mathscr{A}$-modules where $\mathscr{A}$ is the sheaf of algebraic local rings. We propose to compare the sheaves $\mathscr{A}$, $\mathscr{B}$, $\mathscr{O}$. For convenience we shall adopt the following notation; if S is a sheaf of $\mathscr{A}$-modules then we write $S^b = S \otimes_{\mathscr{A}} \mathscr{B}$, $S^0 = S \otimes_{\mathscr{A}} \mathscr{O}$; if S is a sheaf of $\mathscr{B}$-modules then we write $S^0 = S \otimes_{\mathscr{B}} \mathscr{O}$.

PROPOSITION (1.4). *Let X be a (complex) algebraic manifold. Then $S \to S^b$ is an exact functor from the category of coherent sheaves of $\mathscr{A}$-modules to the category of coherent sheaves of $\mathscr{B}$-modules.*

This is proved in [20; Prop. 10].

PROPOSITION (1.5). *Let X be a complex manifold. Then $S \to S^0$ is an exact functor from the category of coherent sheaves of $\mathscr{B}$-modules to the category of coherent sheaves of $\mathscr{O}$-modules.* The fact that S^0 is coherent, if S is coherent, is proved in [5; (2.9)]. For the exactness we observe that the property is a local one, so that it is sufficient to prove that

$$S_x \to S_x \otimes_{\mathscr{B}_x} \mathscr{O}_x$$

is an exact functor. This is proved in (1.7) below as a consequence of the following lemma.

† All manifolds are assumed to have countable topology.

LEMMA (1.6). *Let A, B be (Noetherian) local rings with maximal ideals $m(A)$, $m(B)$ and m-adic completions $\hat{A}$, $\hat{B}$. Let $\phi : A \to B$ be a homomorphism with $\phi(m(A)) \subset m(B)$ and suppose that $\hat{B}$ is a flat† $\hat{A}$-module (the module structure being induced by ϕ). Then B is a flat A-module.*

Proof:‡ We shall prove this using the results and terminology given in [13] (the proofs of these results will appear in [8]). First we have [13; 7.3.5] that $\hat{A}$ is a faithfully-flat A-module and $\hat{B}$ is a faithfully-flat B-module. Then, since $\hat{B}$ is a flat $\hat{A}$-module and $\hat{A}$ is a flat A-module it follows [13; 6.2.1] that $\hat{B}$ is a flat A-module. From this and the fact that $\hat{B}$ is a faithfully-flat B-module we deduce [13; 6.6.4] that B is a flat A-module.

Remark. This lemma gives a proof of the exactness part of (1.4). For this application we actually have $\hat{A} = \hat{B}$.

LEMMA (1.7). *Let $A \subset \mathbf{C}[[z_1, \dots, z_n]]$, $B \subset \mathbf{C}[[z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n]]$ be the subrings consisting of power series convergent in some neighbourhood of the origin. Then B is a flat A-module.*

Proof: A and B are well-known to be Noetherian local rings. Hence, in view of (1.6), it is sufficient to show that $\hat{B} = \mathbf{C}[[z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n]]$ is a flat $\hat{A}$-module, where $\hat{A} = \mathbf{C}[[z_1, \dots, z_n]]$. But $\hat{B}$ is a direct product of free $\hat{A}$-modules, a free module is flat and flatness commutes with direct products [13; 6.2.1]. This completes the proof.

We recall next the original construction of Grothendieck [7]. Let X be a quasi-projective algebraic manifold and let $K^a(X)$ be the Grothendieck group constructed from the algebraic vector bundles on X. Then every coherent sheaf S (of $\mathscr{A}$-modules) on X has a resolution

$$0 \to L_n \to L_{n-1} \to \dots \to L_0 \to S \to 0$$

by locally free sheaves L_i. Moreover, if E_i is the vector bundle associated to L_i, the element

$$\gamma^a(S) = \sum_{i=0}^{n} (-1)^i E_i \in K^a(X)$$

is independent of the resolution and depends only on S.

From (1.4), (1.5) and the definitions of $\gamma(S)$, $\gamma^a(S)$ we obtain the required compatibility:

PROPOSITION (1.8). *Let S be a coherent sheaf of $\mathscr{A}$-modules on a quasi-projective algebraic manifold X. Then we have*

$$\gamma(S^0) = \alpha(\gamma^a(S)),$$

where $\alpha : K^a(X) \to \mathscr{K}(X)$ is the natural homomorphism.

§2. COMPLEX TUBULAR NEIGHBOURHOODS

We start with a result on Stein manifolds.

PROPOSITION (2.1). *Let X be a Stein manifold, Y a closed complex submanifold, N the*

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† We recall that M is a flat A-module if $\otimes_A M$ is an exact functor.

‡ We are indebted to J-P. Serre for this proof.

normal bundle of Y in X. Then there exists an open neighbourhood A of† *Y in N, a neighbourhood B of Y in X and a complex analytic homeomorphism $f\colon A \to B$ which is the identity on Y.*

Proof:‡ By a theorem of Remmert [19] we can embed X in $\mathbf{C}^n$. We then consider the following three holomorphic vector bundles on Y:

P = tangent bundle of Y,

Q = tangent bundle of X restricted to Y,

R = tangent bundle of $\mathbf{C}^n$ restricted to Y.

Let $P' = R/P$, $Q' = R/Q$, then we have the exact sequences

$$0 \to P \to Q \to N \to 0$$

$$0 \to N \to P' \to Q' \to 0\,.$$

Now Y is also a Stein manifold and hence every short exact sequence of holomorphic vector bundles on Y splits [1; Prop. 2]. Thus we get direct sum decompositions:

$$Q = P \oplus N, \qquad P' = N \oplus Q', \qquad R = P \oplus P',$$

from which we deduce

$$R = Q \oplus Q'.$$

Here we have, for simplicity, omitted the splitting maps $N \to Q$ etc. We shall also make the obvious affine identification between $\mathbf{C}^n$ and the tangent space R_y to $\mathbf{C}^n$ at a point y (the origin of R_y corresponding to $y \in \mathbf{C}^n$). Thus R becomes a subspace of $X \times \mathbf{C}^n$. Now, for each point $y \in Y$, we consider the situation in R_y. Then P'_y is transversal to X at y and so intersects X locally in a submanifold X_y. The projection in P'_y parallel to Q'_y gives a local isomorphism of N_y onto X_y, and as y varies we obtain in this way a holomorphic map $f\colon A_0 \to X$ where A_0 is a neighbourhood of Y in N. Moreover f is the identity on Y and has non-vanishing Jacobian. It remains to show that, in some smaller neighbourhood A, f will be globally (1–1). The proof of this is elementary and left to the reader.

Next we give a real-analytic analogue of (2.1) for arbitrary complex manifolds.

PROPOSITION (2.2). *Let X be a complex manifold, Y a closed complex submanifold, N the normal bundle of Y in X. Then there exists an open neighbourhood A of Y in N, a neighbourhood B of Y in X and a real-analytic homeomorphism $f\colon A \to B$ which is the identity on Y and is such that $f|A \cap Ny$ is holomorphic for all $y \in Y$.*

Proof: Let $\bar{X}$ denote X with its conjugate complex structure and put $X_0 = X \times \bar{X}$. Let Δ denote the diagonal in X_0; then Δ is real-analytically homeomorphic to X and X_0 is a complexification of Δ. Then, according to a result of Grauert [10], there is a neighbourhood X_1 of Δ in X_0 which is a Stein manifold. If $p\colon X \times \bar{X} \to X$ is the projection we put:

$$Y_0 = p^{-1}(Y), \qquad Y_1 = X_1 \cap Y_0.$$

† We identify Y with the zero-cross-section of N in the obvious way.
‡ The proofs of (2.1) and (2.2) are due to H. Grauert.

We now apply (2.1) to the pair X_1, Y_1 and obtain a complex-analytic homeomorphism

$$f_1 : A_1 \to B_1,$$

where A_1 is a neighbourhood of Y_1 in N_1 (its normal bundle) and B_1 is a neighbourhood of Y_1 in X_1. But $N_1 \cong p^{-1}(N)$, and so the restriction of N_1 to $Y \cong \Delta \cap Y_1$ may be identified (as a real-analytic complex vector bundle) with N. The part A' of A_1 over Y is then a neighbourhood of Y in N and $f = pf_1 : A' \to X$ is a real-analytic map which is the identity on Y, has non-vanishing Jacobian on Y and is holomorphic on each fibre $A' \cap N_y$. As in (2.1) we now find a smaller neighbourhood A on which f is (1–1) and this completes the proof.

Consider next a real-analytic complex vector bundle E over a real-analytic manifold Y. Let $(z_1, \dots, z_n)$ be the fibre co-ordinates given by a local product decomposition. If $(z_1', \dots z_n')$ is another system of fibre co-ordinates we have equations

$$z_i = \sum_{j=1}^{n} a_{ij} z_j'$$

where the a_{ij} are real-analytic functions defined locally on Y, and $\det(a_{ij}) \neq 0$. Thus $z_1, \dots, z_n$ and $z_1', \dots, z_n'$ generate locally the same sheaf of ideals of $\mathcal{O}_E$. In this way we get a sheaf of ideals of $\mathcal{O}_E$ defined globally; we shall denote this sheaf by $J(E)$.

Returning now to the situation of (2.2) we let $I(Y)$ denote the sheaf of ideals of $\mathcal{B}_X$ consisting of germs which vanish on Y; as in §1 this defines a sheaf of ideals $I(Y)^0$ of $\mathcal{O}_X$.

PROPOSITION (2.3). *In the situation of* (2.2) *the sheaf* $J(N)$ *corresponds under* f *to the sheaf* $I(Y)^0$.

Proof: For points not on Y the result is obvious. For a point $y \in Y$ we take local complex co-ordinates $(w_1, \dots, w_m)$ on X with Y given by $w_1 = \dots = w_n = 0$; on N we take local real co-ordinates $u_1, \dots, u_{2m-2n}$ on Y and complex fibre co-ordinates $z_1, \dots, z_n$. Then f is given by equations

$$w_i = f_i(z_1, \dots, z_n; u_1, \dots, u_{2m-2n}),$$

where the f_i are analytic; the fact that the f_i do not involve the $\bar{z}_j$ follows from the last assertion of (2.2). Since $z_1 = \dots = z_n = 0$ corresponds to $w_1 = \dots = w_n = 0$ it follows that, for $1 \leqslant i \leqslant n$, f_i belongs to the ideal generated by $z_1, \dots, z_n$. Since moreover f is a real-analytic homeomorphism we have

$$\det\left(\frac{\partial f_i}{\partial z_j}\right) \neq 0$$

where $1 \leqslant i, j \leqslant n$, and so $f_1, \dots, f_n$ actually generate the same ideal as $z_1, \dots, z_n$. Since $w_1, \dots, w_n$ are local generators for $I(Y)^0$ and $z_1, \dots, z_n$ are local generators for $J(N)$ this establishes the proposition.

From the exact sequence of sheaves of $\mathcal{B}_X$-modules

$$0 \to I(Y) \to \mathcal{B}_X \to \mathcal{B}_Y \to 0$$

we get, using (1.5), the exact sequence of sheaves of $\mathcal{O}_X$-modules

$$0 \to I(Y)^0 \to \mathcal{O}_X \to (\mathcal{B}_Y)^0 \to 0.$$

Hence, if we define $S(N) = \mathcal{O}_N/J(N)$, Proposition (2.3) shows that $S(N)$ corresponds under f to $(\mathscr{B}_Y)^0$. More generally, let E be a holomorphic vector bundle on Y, L the associated locally free sheaf of $\mathscr{B}_Y$-modules (extended by zero to $X - Y$) and let L' be the locally free sheaf of $\mathcal{O}_N$-modules associated to $p^*(E)$, where $p : N \to Y$ is the projection map. Then each local isomorphism $\alpha : E \to Y \times \mathbf{C}^p$ gives a local isomorphism $\beta : f^*(L^0) \to S(N) \otimes L'$. It is easy to check that β is in fact independent of α. Thus we have:

PROPOSITION (2.4). *In the situation of* (2.2), *let E be a holomorphic vector bundle on Y, L the associated sheaf, L' the sheaf associated to $p^*(E)$. Then $S(N) \otimes L'$ corresponds under f to L^0.*

The advantage of the preceding results is that the sheaf $S(N)$ has an explicit resolution obtained as follows. The real-analytic complex vector bundle $p^*(N)$ over the real-analytic manifold N has a natural cross section s ("the diagonal"). If $z_1, \dots, z_n$ are local fibre co-ordinates in N then s is given locally by the n functions $z_1, \dots, z_n$. Since these obviously have property (P) of [5; §2], and since the sheaf of zeros of s [5; §2] coincides with the sheaf $S(N)$ we deduce from [5; (2.13)] the following result:

PROPOSITION (2.5). *Let N be a real-analytic complex vector bundle over a real-analytic manifold Y, with projection map p. Then the sheaf $S(N)$ defined above has a resolution:*

$$0 \to L_n \xrightarrow{\alpha_n} L_{n-1} \to \dots \xrightarrow{\alpha_1} L_0 \xrightarrow{\varepsilon} S(N) \to 0$$

where L_i is the locally free sheaf corresponding to the vector bundle $\lambda^i(p^*(N^*))$, α_i is given by the interior product with the diagonal section s of $p^*(N^*)$, ε is the natural map of $L_0 = \mathcal{O}_N$ onto $\mathcal{O}_N/J(N) = S(N)$ and $n = \dim N$.

In view of (2.2) and (2.3) we can transfer the resolution of $S(N)$ given by (2.5) to some neighbourhood of Y in X, giving a resolution of the sheaf $(\mathscr{B}_Y)^0$. This resolution will be used in the next section.

§3. THE RIEMANN–ROCH THEOREM FOR EMBEDDINGS

We consider now the following problem. Let X be a complex manifold, Y a closed complex submanifold, E a holomorphic vector bundle on Y and let L be the corresponding locally free sheaf of $\mathscr{B}_Y$-modules (regarded as a sheaf extended by zero on $X - Y$). We write $\gamma_Y(E)$ for the element $\gamma_Y(L^0) \in \mathscr{K}(X, X - Y)$, and $\gamma(E)$ for $\gamma(L^0) \in \mathscr{K}(X)$. The problem then is to calculate the Chern classes and Chern character of $\gamma(E)$ in terms of those of E. The answer is provided by the following theorem, whose proof will occupy most of this section.

THEOREM (3.1). *In the notation above we have*

(i) $\operatorname{ch}(\gamma(E)) = i_*(\mathfrak{T}(N)^{-1}\operatorname{ch}(E))$,

(ii) $c(\gamma(E)) = i_*\left(\dfrac{c(\lambda_{-1}(N^*))}{c_n(N)} * c(E)\right)$,

where N is the normal bundle of Y in X, $n = \dim N$, $\mathfrak{T}$ is the Todd polynomial, $i_* : H^*(Y) \to H^*(X)$ is the Gysin homomorphism, N^* is the dual of N, $\lambda_{-1} = \sum_{i=0}^{n} (-1)^i \lambda^i$ and $a * b$ is the universal formula giving the Chern class of a tensor product.

Remarks. (1) The precise definition of the operation $*$ is as follows. Let A be a graded commutative ring with unit and let A' denote the subset of elements of A of the form $(1 + \text{higher terms})$. If $a, b \in A'$ we write formally

$$a = \prod_i (1 + x_i), \qquad b = \prod_j (1 + y_j),$$

where the x_i, y_j have formal weight one. Then $a * b \in A'$ is defined by putting

$$a * b = \prod_{i,j} (1 + x_i + y_j).$$

This is to be understood in the following sense; we expand the right-hand side in terms of the elementary symmetric functions of the x_i and of the y_j and then substitute the components of a, b. In (ii) it is implied that the term $c(\lambda_{-1}(N^*))/c_n(N)$ is in fact of the form $(1 + \text{higher terms})$.

(2) If X, Y are complex quasi-projective algebraic manifolds this theorem is essentially the Grothendieck Riemann–Roch theorem for embeddings. Formula (i) is proved in [7; Prop. 15] and (i) and (ii) are proved in [12; Chap. 2]. However, the following points should be noted. Our formulae hold in $\mathscr{H}^*(X; \mathbf{Q})$ and $\mathscr{H}^*(X; \mathbf{Z})$ whereas in Grothendieck's case formula (i) holds in the Chow ring (with rational coefficients) and hence in $H^*(X; \mathbf{Q})$ and formula (ii) holds in $GK^a(X)$, the graded ring associated to $K^a(X)$. The difference between $\mathscr{H}^*$ and H^* is probably only an apparent one since it is probable that an algebraic manifold has the homotopy type of a finite polyhedron: this would follow for instance if we assumed the triangulability of pairs of algebraic varieties (with arbitrary singularities). For the second point let us for simplicitly assume that X has in fact the homotopy type of a finite polyhedron —for example X could be compact. Then it can be shown that the natural homomorphism

$$K^a(X) \to K(X)$$

is a homomorphism of filtered groups, and hence induces a homomorphism of graded groups

$$GK^a(X) \to GK(X).$$

On the other hand we have [4; (2.1)] the spectral sequence

$$H^*(X; \mathbf{Z}) \Rightarrow K^*(X).$$

Moreover it is known that this spectral sequence is not trivial for all algebraic manifolds [2; §6] so that it is not possible to deduce (3.1)(ii) from Grothendieck's result. In [17] Porteous quotes Grothendieck though he in fact requires our result (3.1)(ii). As remarked in the introduction his results can now, using (3.1), be extended to arbitrary complex manifolds.

To prove (3.1) we shall first show that $\gamma(E)$ can be defined in a more general situation than the one so far discussed; in particular no analytic structure is required.

Let X be a differentiable manifold, Y a closed differentiable submanifold of codimension $2n$. We suppose that the normal bundle of Y in X is given an almost complex structure, and we denote the complex vector bundle so obtained by N. Let E be any (topological) complex vector bundle on Y. We shall now define elements $\gamma_Y^t(E) \in \mathscr{K}(X, X - Y)$ and $\gamma^t(E) \in \mathscr{K}(X)$;

the superscript ^{t} stands here for "topological". Using a Riemannian metric on X we can find an open neighbourhood N_0 of Y in N, an open neighbourhood X_0 of Y in X and a homeomorphism $f: N_0 \to X_0$ which is the identity on Y. Then, according to [5; (2.12)] we have a sequence of vector bundles

$$0 \to F_n \xrightarrow{\alpha_n} F_{n-1} \to \ldots \xrightarrow{\alpha_1} F_0 \to 0$$

defined on X_0 and exact on $X_0 - Y$, where $F_i = p^*(\lambda^i(N^*))$, $p: X_0 \to Y$ corresponds to the bundle projection $N \to Y$ and α_i is the interior product with the "diagonal section". From this we get the generalized difference bundle (5; §3),

$$d(F_i, \alpha_i) \in \mathscr{K}(X_0, X_0 - Y).$$

Since $\mathscr{K}(X_0, X_0 - Y)$ is a $\mathscr{K}(X_0)$-module we can form the product

$$p^*(E).d(F_i, \alpha_i) \in \mathscr{K}(X_0, X_0 - Y).$$

Finally, using the excision isomorphism (1.2),

$$\sigma: \mathscr{K}(X_0, X_0 - Y) \to \mathscr{K}(X, X - Y),$$

we define

$$\gamma_Y^t(E) = \sigma(p^*(E).d(F_i, \alpha_i)).$$

The image of $\gamma_Y^t(E)$ in the natural homomorphism $\mathscr{K}(X, X - Y) \to \mathscr{K}(X)$ will be denoted by $\gamma^t(E)$.

It is not difficult to show that this definition is independent of the choice of X_0 and of f. Since however this fact plays no essential role in this paper we shall omit its proof.

We can now show that γ^t is indeed a generalization of γ. If E is a holomorphic or real-analytic vector bundle we shall, when emphasis is required, denote by E^t the underlying topological vector bundle.

THEOREM (3.2). *Let X be a complex manifold, Y a closed complex submanifold, E a holomorphic vector bundle on Y. Then*

(i) $\gamma_Y(E) = \gamma_Y^t(E^t) \in \mathscr{K}(X, X - Y)$

(ii) $\gamma(E) = \gamma^t(E^t) \in \mathscr{K}(X)$.

Proof: We first observe that (ii) follows from (i) on applying the natural homomorphism $\mathscr{K}(X, X - Y) \to \mathscr{K}(X)$. To prove (i) it is sufficient, in view of (1.2), to replace X by any neighbourhood X_0 of Y in X. Let f, A, B be as given by (2.2) and take $N_0 = A$, $X_0 = B$. Using (2.2) and (2.4) we see that the problem can be transferred to A and that it is sufficient to prove that

$$\gamma_Y(S(N) \otimes L') = \gamma_Y^t(E^t) \in \mathscr{K}(A, A - Y)$$

But by (1.3) we have

$$\gamma_Y(S(N) \otimes L') = \gamma_Y(S(N)).p^*(E),$$

and by definition

$$\gamma_Y^t(E^t) = \gamma_Y^t(1).p^*(E).$$

Finally, from (2.5) and the definition of $\gamma_Y^!(1)$, we see that $\gamma_Y(S(N))$ and $\gamma_Y^!(1)$ are both given by the same generalized difference bundle $d(F_i, \alpha_i)$ with $F_i = p^*(\lambda^i(N^*))$. This shows that $\gamma_Y(S(N)) = \gamma_Y^!(1)$ and completes the proof.

In view of (3.2), Theorem (3.1) will follow from

THEOREM (3.3). *Let X be a differentiable manifold, Y a closed differentiable submanifold with almost complex normal bundle N, and let E be a complex vector bundle on Y. Then*

(i) $\operatorname{ch}(\gamma^!(E)) = i_*(\mathfrak{T}(N)^{-1}.\operatorname{ch}(E))$,

(ii) $$c(\gamma^!(E)) = i_*\left(\frac{c(\lambda_{-1}(N^*))}{c_n(N)} * c(E)\right),$$

where $n = \dim N$.

Proof: We shall in fact prove a stronger version than stated, namely we shall prove (i) and (ii) for $\gamma_Y^!(E)$. The required formulae will then follow using the homomorphisms induced by the map $(X, \emptyset) \to (X, X - Y)$. Now, in view of the definition of $\gamma_Y^!(E)$ and the fact that $i_* : \mathscr{H}^*(Y) \to \mathscr{H}^*(X, X - Y)$ is an $\mathscr{H}^*(Y)$-homomorphism, it is sufficient to prove (i) and (ii) for $\gamma_Y^!(1)$. Then, from the functorial nature of the definition of $\gamma_Y^!(1)$, it is sufficient to consider the universal case, i.e. we may take Y to be an approximation to $B_{U(n)}$ and $X = N$ to be the standard vector bundle over it. But, in this case,

$$j^* : \mathscr{H}^*(X, X - Y) \to \mathscr{H}^*(X),$$

with coefficients $\mathbf{Z}$ or $\mathbf{Q}$, maps the first group isomorphically onto the ideal generated by c_n (up to the approximating dimension). Hence it is sufficient to prove (i) and (ii) in $\mathscr{H}^*(X)$. But, by a property of the generalized difference bundle [5; (3.4)(iv)], we have

$$j^*d(p^*\lambda^i(N^*), \alpha_i) = p^*\left(\sum_{i=0}^{n} (-1)^i\lambda^i(N^*)\right).$$

Also $p : X \to Y$ is a homotopy equivalence and $p^*j^*i_*$ is just the multiplication by c_n. Thus (i) follows from the formula [7; Lemma 18]

$$\operatorname{ch} \lambda_{-1}(N^*) = c_n.\mathfrak{T}(N)^{-1},$$

and (ii) is immediate.

§4. THE GROTHENDIECK RIEMANN-ROCH THEOREM

In this section we shall restrict ourselves to *compact* manifolds X, so that $\mathscr{K}(X) \cong K(X)$. We shall indicate how the results of §3 can be extended from embeddings to general maps, giving in particular a proof of GRR for (projective) complex algebraic manifolds.

Let X be a projective algebraic manifold and let $K^a(X)$ denote as before the Grothendieck group constructed from algebraic vector bundles over X. Then for a morphism

$$f: Y \to X$$

of projective manifolds, Grothendieck [7] has defined a "direct image" homomorphism

$$f_!^a : K^a(Y) \to K^a(X).$$

This has the following basic properties [7]:

(T) $(fg)_!^a = f_!^a g_!^a$

(M) $f_!^a(y \,.\, f_a^!(x)) = f_!^a(y).x$,
for $x \in K^a(X)$, $y \in K^a(Y)$ and $f_a^! : K^a(X) \to K^a(Y)$ the natural homomorphism,

(P) $(f \times i)_!^a(y \otimes 1) = f_!^a(y) \otimes 1$,
where $y \in K^a(Y)$, $i : Z \to Z$ is the identity and $f \times i : Y \times Z \to X \times Z$ is the product map.

We shall now show how to define an analogous homomorphism

$$f_! : K(Y) \to K(X)$$

for any continuous map f: $Y \to X$, where Y, X are almost-complex manifolds. In fact, as indicated in [3], $f_!$ can be defined in even more general circumstances but for our present purposes the almost-complex case is sufficient. The verifications of the various statements which we shall make concerning the basic properties of $f_!$ will not be given here but will appear in a future publication. Our aim here is not to develop the topological theory in its own right but to connect it up with the analytic theory.

Let $t(X)$, $t(Y)$ denote the tangent bundles of X and Y. Let n be a large integer and let F: $Y \to X \times \mathbf{C}^n$ be a differentiable embedding which is homotopic to the map $y \to f(y) \times 0$. The normal bundle of $F(Y)$ in $X \times \mathbf{C}^n$ can then be given a unique almost complex structure N such that the following equation holds in $K(Y) = K(F(Y))$:

$$N = f^! t(X) + n - t(Y).$$

Hence if E is any complex vector bundle on Y we may apply the construction of §3 and obtain an element

$$\gamma^t(E) \in \mathscr{K}(X \times \mathbf{C}^n, X \times \mathbf{C}^n - F(Y)).$$

Since $\gamma^t(E)$ is additive in E this extends to a homomorphism

(1) $$K(Y) \to \mathscr{K}(X \times \mathbf{C}^n, X \times \mathbf{C}^n - F(Y)).$$

Now let $a \in S^{2n}$ be a base point and identify $\mathbf{C}^n$ with $S^{2n} - \{a\}$. Since $F(Y)$ is compact we may regard it as embedded in $X \times S^{2n}$. Thus using excision we get a natural homomorphism,

(2) $$\mathscr{K}(X \times \mathbf{C}^n, X \times \mathbf{C}^n - F(Y)) \to \mathscr{K}(X \times S^{2n}, X \times a).$$

Since X is a compact manifold we have

(3) $$\mathscr{K}(X \times S^{2n}, X \times a) \cong K(X \times S^{2n}, X \times a).$$

Finally, and most important, we have the Bott isomorphism [4; (1.7)]

(4) $$K(X \times S^{2n}, X \times a) \cong K(X).$$

Combining (1)—(4) we obtain the required homomorphism

$$f_! : K(Y) \to K(X).$$

It can be shown that $f_!$ is independent of all choices made and depends in fact only on the homotopy type of f. Moreover properties (T), (M) and (P) can also be shown to hold.

Since the Chern character commutes with the Bott isomorphism (4) [4; (1.10)] the following theorem follows at once from the definition of $f_!$ and (3.3)(i).

THEOREM (4.1). *Let Y, X be almost-complex manifolds, $f: Y \to X$ a continuous map. Then, for any $y \in K(Y)$, we have*

$$\mathfrak{T}(X).\mathrm{ch}\, f_!(y) = f_*(\mathfrak{T}(Y).\mathrm{ch}\, y).$$

Remark. This is similar to Theorem 1 of [3], although here we have claimed (but not proved) that $f_!$ is functorially defined.

GRR, for the complex projective case, will then follow from (4.1) and the following theorem.

THEOREM (4.2). *Let Y, X be projective complex algebraic manifolds, $f: Y \to X$ a morphism. Then we have a commutative diagram:*

$$\begin{array}{ccc} K^a(Y) & \xrightarrow{f_!^a} & K^a(X) \\ \downarrow & & \downarrow \\ K(Y) & \xrightarrow{f_!} & K(X) \end{array}$$

Remark. Theorem (4.2) is, in a sense, more precise than GRR since it gives the homotopy type of $f_!^a(y)$ as opposed to its cohomology type.

Since $f: Y \to X$ can be factored in the form†

$$Y \xrightarrow{i} P_n \times X \xrightarrow{p} X,$$

where i is an embedding and p is the projection, and since both $f_!$ and $f_!^a$ satisfy (T), it is sufficient to consider separately the cases $f = i$, $f = p$. The case of an embedding is dealt with by (1.8) and (3.2), so that it remains to consider p. For this we require the following facts proved in [7; Props. 7 and 9].

LEMMA (4.3). *Let X be a projective algebraic manifold, then*

(i) *$K^a(P_n) \otimes K^a(X) \to K^a(P_n \times X)$ is an epimorphism,*

(ii) *$K^a(P_{n-1}) \xrightarrow{i_!^a} K^a(P_n) \xrightarrow{\varepsilon} \mathbf{Z}$ is exact, where ε denotes the augmentation and $i: P_{n-1} \to P_n$ is the inclusion.*

We are now ready to prove (4.2) for the projection $p: P_n \times X \to X$. In view of (4.3)(i) and the fact that $f_!$ and $f_!^a$ both satisfy (M) it is sufficient to prove (4.2) for elements $y \otimes 1$ with $y \in K^a(P_n)$. Then, since $f_!$ and $f_!^a$ both satisfy (P), it is sufficient to prove (4.2) for the map $f: P_n \to P_0$, where P_0 is a point. Now using (4.3)(ii), induction on n and the fact that $f_!$ and $f_!^a$ both satisfy (T) we see that it is sufficient to prove (4.2) for $f: P_n \to P_0$ and $y = 1$. Now for a point P_0 we may identify $K^a(P_0)$ and $K(P_0)$ with $\mathbf{Z}$ and then, by definition,

$$f_!^a(1) = \sum_{q=0}^{n} (-1)^q \dim H^q(P_n, \mathcal{O}),$$

† P_n denotes complex projective n-space.

where $\mathcal{O}$ is the fundamental sheaf on P_n. But by [21; p. 275]

$$H^q(P_n, \mathcal{O}) = 0 \qquad \text{for } q > 0,$$
$$H^0(P_n, \mathcal{O}) = \mathbf{C}.$$

Thus $f_!^q(1) = 1$. Finally therefore we have to prove that $f_!(1) = 1$, or equivalently that $\operatorname{ch} f_!(1) = 1$. But this follows from (4.1) and the fact that $f_*\mathfrak{T}(P_n)$ (the Todd genus of P_n) is the coefficient of x^n in $((1 - e^{-x})/x)^{-n-1}$ [14; (1.7.1)]. This completes the proof of (4.2).

§5. REAL BUNDLES

So far we have considered only complex vector bundles and complex-valued functions. It is however possible to give parallel results in the real case. Instead of Chern classes we now have Stiefel–Whitney (mod 2) classes and Pontrjagin (rational) classes. For reasons which will become apparent we cannot deal effectively with the integral Pontrjagin classes. Since the Pontrjagin classes are, up to sign, the Chern classes of the complexification, the rational Pontrjagin classes can effectively be replaced by the Chern character of the complexification. The appropriate analogue of (3.1) is then

THEOREM (5.1). *Let X be a real-analytic manifold, Y a real-analytic submanifold with normal bundle N of dimension n. Let E be a real-analytic real vector bundle on Y and let $\gamma(E) \in \mathscr{K}\mathcal{O}(X, X - Y)$ be the "real Grothendieck element" constructed by means of resolutions of sheaves as in* [5; §4], *but with the word "complex" replaced everywhere by "real". Then*

(i) $$w(\gamma(E)) = i_*\left\{\frac{w(\lambda_{-1}(N))}{w_n(N)} * w(E)\right\}.$$

(ii) *If N is orientable*

$$\operatorname{ch} \gamma(E) = (-1)^k i_*(1) . i_*(\mathfrak{T}(N)^{-1} \operatorname{ch} E) \qquad (n = 2k)$$
$$= 0 \quad (n \text{ odd})$$

Remarks. (1) w denotes the total Stiefel–Whitney class (mod 2).

(2) For a real vector bundle F we define $\mathfrak{T}(F)$, ch (F) to be $\mathfrak{T}$ and ch of the complexification of F.

(3) To define the Gysin homorphism $i_* : H^*(Y) \to H^*(X)$ it is necessary in (ii) to orient N. The result is however independent of the choice of orientation. Note that in the mod 2 case i_* is defined without any restriction on N.

The proof of (5.1) is similar to that of (3.1). For (i) we use the fact (cf. [6]) that the kernel of

$$H^*(B_{0(n)}; \mathbf{Z}_2) \to H^*(B_{0(n-1)}; \mathbf{Z}_2)$$

is the ideal (w_n). For (ii) we use the fact (cf. [6]) that the kernel of

$$H^*(B_{SO(n)}; \mathbf{Q}) \to H^*(B_{SO(n-1)}; \mathbf{Q})$$

is zero for n odd and is the ideal generated by the Euler class X_k if $n = 2k$. The extra factor $i_*(1)$ in (ii) arises from the formula

$$\begin{aligned}\operatorname{ch} \lambda_{-1}(N) &= \prod_{i=1}^{k} (1 - e^{x_i})(1 - e^{-x_i}) \\ &= (-1)^k \left(\prod x_i\right)^2 \prod \left(\frac{1 - e^{x_i}}{x_i}\right) \cdot \prod \left(\frac{1 - e^{-x_i}}{-x_i}\right) \\ &= (-1)^k X_k(N)^2 \,.\, \mathfrak{T}(N)^{-1};\end{aligned}$$

on passing from the bundle space to the Thom complex only one of the factors $X_k(N)$ is absorbed. Formula (ii) can also be written

$$\operatorname{ch} \gamma(E) = (-1)^k i_*(X_k(N) \,.\, \mathfrak{T}(N)^{-1} \,.\, \operatorname{ch}(E)).$$

This shows that $\operatorname{ch} \gamma(E) = 0$ if $X_k(N) = 0$. In particular this always holds in the stable case ($k > \dim Y$) so that the interest of (ii) lies only in the unstable case. There is thus no analogue of (4.1) on these lines.†

Using the methods of [17] Theorem (5.1) gives the behaviour of Stiefel–Whitney and Pontrjagin classes under "blowing up" in the real-analytic case. Also, as in [18], (5.1) can be applied to give information on the Thom polynomials of mappings of real-analytic manifolds.

§6. THE DIFFERENTIABLE CASE

The proof of GRR given in §4 was heavily analytic. It rested essentially on the main results of the theory of coherent analytic sheaves. In this section we shall give a brief indication of an alternative approach which avoids this theory, and uses sheaves of C^∞ functions instead. We shall obtain a C^∞ analogue of GRR which, in the real case, is a more appropriate theorem for the applications considered in [17, 18].

We consider a compact C^∞-differentiable manifold X and the sheaf $\mathscr{D}$ of germs of C^∞ maps $X \to \mathbf{C}$. We shall deal with sheaves of $\mathscr{D}$-modules and for convenience we introduce the following definition.

DEFINITION (6.1). *A sheaf S of $\mathscr{D}$-modules on X is called coherent if, in some neighbourhood of its support, it has a (finite) resolution by locally free sheaves (of finite rank).*

Since any sheaf S of $\mathscr{D}$-modules is fine we have $H^1(X, S) = 0$. Unlike "Theorem B" this is an elementary result (cf. [9; (4.4.3)]). The analogues of [5; (2.7) and (2.8)] therefore hold for $\mathscr{D}$-modules, and hence as in §4 we can associate to each coherent sheaf of $\mathscr{D}$-modules on X, with support in Y, an element of $\mathscr{K}(X, X - Y)$. This element will be denoted by $\gamma^d(S)$.

Now we have the following important result of Malgrange.

† In [3] an alternative approach to the real case is shown to yield a satisfactory analogue of (4.1). This approach does not on the other hand have any connection with sheaves.

PROPOSITION (6.2). *The functor $S \to S \otimes_{\mathcal{O}} \mathcal{D}$ is exact on the category of coherent sheaves of $\mathcal{O}$-modules.*

Actually Theorem 2 of [16; IV] deals only with real-valued functions. The extension to complex-valued functions is quite elementary and can be proved as follows. Let $\mathcal{O}'$, $\mathcal{O}$ denote the germs at the origin of real-analytic maps $\mathbf{R}^n \to \mathbf{R}$ and $\mathbf{R}^n \to \mathbf{C}$ respectively; let $\mathcal{D}'$, $\mathcal{D}$ be defined similarly using C^∞ maps. Malgrange's theorem asserts that $\mathcal{D}'$ is $\mathcal{O}'$-flat. Then by [13; (6.2.1)] it follows that $\mathcal{D}' \otimes_{\mathcal{O}'} \mathcal{O}$ is $\mathcal{O}$-flat. But

$$\mathcal{D}' \otimes_{\mathcal{O}'} \mathcal{O} \cong \mathcal{D}' \otimes_{\mathcal{O}'} (\mathcal{O}' \otimes_R \mathbf{C}) \cong \mathcal{D}' \otimes_R \mathbf{C} \cong \mathcal{D}.$$

Thus $\mathcal{D}$ is $\mathcal{O}$-flat which is the assertion of (6.2).

As a corollary of (6.2) we have

COROLLARY (6.3). *If S is a coherent sheaf of $\mathcal{O}$-modules then $S \otimes_{\mathcal{O}} \mathcal{D}$ is a coherent sheaf of $\mathcal{D}$-modules.*

Remark. This corollary provides some justification for definition (6.1) and guarantees the existence of many coherent sheaves of $\mathcal{D}$-modules. In particular the resolution of (2.5) goes over to the C^∞-case (a fact which could be checked directly).

From (1.4), (1.5) and (6.3) we deduce

PROPOSITION (6.4). *Let X be a complex algebraic manifold. Then the functor $S \to S \otimes_{\mathcal{A}} \mathcal{D}$ is an exact functor from the category of coherent sheaves of $\mathcal{A}$-modules to the category of coherent sheaves of $\mathcal{D}$-modules.*

From this point on the proof of GRR proceeds as before except that we still need the C^∞-analogue of (2.2). Of course this follows from (2.2) itself but if we want a treatment independent of Theorems A and B then a direct proof has to be sought. It is probable that a proof can be constructed on the lines of [15].

In the real case we can also introduce coherent sheaves of C^∞ functions and obtain the C^∞ analogue of (5.1). There are only two points that require comment. In the first place the real C^∞ analogue of (2.2) is elementary, so that there is no difficulty in this direction. Secondly we have to prove the exactness of the real C^∞ analogue of the sequence of (2.5). This can be done directly, but we can also argue as follows. Since the problem is local we can impose an analytic structure on the C^∞-structure. The exactness of the real-analytic analogue of (2.5) is proved exactly as in (2.5), and we can then apply the theorem of Malgrange (as in (6.2)) to deduce the exactness in the C^∞-case.

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