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Séminaire N. Bourbaki, 1966-1968, exp. nº 314, p. 13-32.
<http://www.numdam.org/item?id=SB_1966-1968__10__13_0>

Séminaire BOURBAKI
19e année, 1966/67, n°314 Novembre 1966

SINGULARITIES AND EXOTIC SPHERES

by Friedrich HIRZEBRUCH

BRIESKORN has proved [4] that the n-dimensional affine algebraic variety $z_o^3 + z_1^2 + \dots + z_n^2 = 0$ (n <u>odd</u>, $n \geq 1$) is a topological manifold though the variety has an isolated singular point (which is normal for $n \geq 2$). Such a phenomenon cannot occur for normal singularities of 2-dimensional varieties, as was shown by MUMFORD ([12], [6]). BRIESKORN's result stimulated further research on the topology of isolated singularities (BRIESKORN [5], MILNOR [11] and the speaker [5], [7]). BRIESKORN [5] uses the paper of F. PHAM [14], whereas the speaker studied certain singularities from the point of view of transformation groups using results of BREDON ([2], [3]), W.C. HSIANG and W.Y. HSIANG [8] and JÄNICH [9].

§ 1. <u>The integral homology of some affine hypersurfaces</u>.

PHAM [14] studies the non-singular subvariety $V_a = V(a_o, a_1, \dots, a_n)$ of $\mathbb{C}^{n+1}$ given by

$$z_o^{a_o} + z_1^{a_1} + \dots + z_n^{a_n} = 1 \qquad (n \geq 0),$$

where $a = (a_o, \dots, a_n)$ consists of integers $a_j \geq 2$.

Let G_{a_j} be the cyclic group of order a_j multiplicatively written and generated by w_j. Define the group $G_a = G_{a_o} \times G_{a_1} \times \dots \times G_{a_n}$ and put $\varepsilon_j = \exp(2\pi i/a_j)$.

Then $w_o^{k_o} w_1^{k_1} \dots w_n^{k_n}$ is an element of G_a whereas $\varepsilon_o^{k_o} \varepsilon_1^{k_1} \dots \varepsilon_n^{k_n}$ is a complex number. G_a operates on V_a by

$$w_o^{k_o} \dots w_n^{k_n}(z_o, \dots, z_n) = (\varepsilon_o^{k_o} z_o, \dots, \varepsilon_n^{k_n} z_n).$$

Let $\hat{G}_{a_j}$ be the group of a_j-th roots of unity and $x \longmapsto \hat{x}$ the isomorphism $G_{a_j} \to \hat{G}_{a_j}$ given by $w_j \longmapsto \varepsilon_j = \hat{w}_j$.

PHAM considers the following subspace U_a of V_a

$$U_a = \{z \mid z \in V_a \text{ and } z_j^{a_j} \text{ real} \geqq 0 \text{ for } j = 0, \dots, n\}$$

LEMMA.- <u>The subspace</u> U_a <u>is a deformation retract of</u> V_a <u>by a deformation compatible with the operations of</u> G_a.

For the proof see PHAM [14], p. 338.

U_a can also be described by the conditions

$$z_j = u_j |z_j| \quad \text{with} \quad u_j \in \hat{G}_{a_j} \qquad (j = 0, \dots, n).$$

Put $|z_j|^{a_j} = t_j$. Then U_a becomes the space of $(n+1)$-tpls of complex numbers

$$t_o u_o \oplus t_1 u_1 \oplus \dots \oplus t_n u_n$$

with

$$u_j \in \hat{G}_{a_j}, \; t_j \geqq 0, \; \sum_{j=0}^{n} t_j = 1$$

Thus U_a can be identified with the join $G_{a_o} * G_{a_1} * \dots * G_{a_n}$ of the finite sets G_{a_j} (see MILNOR [10]).

LEMMA 2.1 in [10] states in particular that the reduced integral homology groups of the join $A * B$ of two spaces A, B without torsion are given by a canonical isomorphism

$$\tilde{H}_{r+1}(A * B) \cong \sum_{i+j=r} \tilde{H}_i(A) \otimes \tilde{H}_j(B),$$

whereas LEMMA 2.2 in [10] shows that $A * B$ is simply connected provided B is arcwise connected and A is any non-vacuous space. These properties of the join together with its associativity imply

THEOREM. <u>The subvariety</u> V_a <u>of</u> $\mathbb{C}^{n+1}$ <u>is</u> $(n-1)$-<u>connected</u>. <u>Moreover</u>

(1) $$\tilde{H}_n(V_a) \cong \tilde{H}_o(G_{a_o}) \otimes \tilde{H}_o(G_{a_1}) \otimes \dots \otimes \tilde{H}_o(G_{a_n}).$$

<u>This is a free abelian group of rank</u> $r = \prod(a_j - 1)$.

<u>The isomorphism</u> (1) <u>is compatible with the operations of</u> G_a.

<u>All other reduced integral homology groups of</u> V_a <u>vanish</u>.

It can be shown that V_a has the homotopy type of a connected union $S^n \vee \dots \vee S^n$ of r spheres of dimension n.

The identification of U_a with a join was explained to the speaker by MILNOR.

$U_a = G_{a_o} * G_{a_1} * \dots * G_{a_n}$ is an n-dimensional simplicial complex which has an n-simplex for each element of G_a. The n-simplex belonging to the unit of G_a is denoted by e. All other n-simplices are obtained from e by operations of G_a. Thus we have for the n-dimensional simplicial chain group

(2) $$C_n(U_a) = J_a\, e$$

where J_a is the group ring of G_a. The homology group $\tilde{H}_n(U_a) = \tilde{H}_n(V_a)$ is an additive subgroup of $J_a\, e = C_n(U_a) \cong J_a$.

The face operator ∂_j commutes with all operations of G_a on $C_n(U_a)$ and furthermore satisfies $\partial_j = w_j \partial_j$. Therefore

(3) $$h = (1-w_o)(1-w_1)\dots(1-w_n)\, e$$

is a cycle. Thus $h \in \tilde{H}_n(U_a)$. It follows easily that $\tilde{H}_n(V_a) = J_a\, h$. This yields the

THEOREM. *The map* $w \to wh$ $(w \in G_a)$ *induces an isomorphism*

$$J_a/I_a \cong \tilde{H}_n(V_a) = J_a\, h$$

where $I_a \subset J_a$ *is the annihilator ideal of* h *which is generated by the elements*

$$1 + w_j + w_j^2 + \ldots + w_j^{a_j-1}\ , \quad (j = 0, \ldots, n).$$

Therefore $w_0^{k_0} w_1^{k_1} \ldots w_n^{k_n} h$ (where $0 \leq k_j \leq a_j-2$, $j = 0,\ldots,n$) *is a basis of* $\tilde{H}_n(V_a)$.

We recall that $\tilde{H}_n(V_a)$ is the integral singular homology group (of course with *compact* support). V_a is a $2n$-dimensional oriented manifold without boundary (non-compact for $n \geq 1$). Therefore the bilinear intersection form S is well defined over $\tilde{H}_n(V_a)$. It is symmetric for n even, skew-symmetric for n odd. It is compatible with the operations of G_a.

PHAM ([14], p.358) constructs an n-dimensional cycle $\tilde{h}$ in V_a which is homologous to h and intersects U_a exactly in two interior points of the sinplices e and $w_0 w_1 \ldots w_n\, e$ (sign questions have to be observed). In this way he obtains (using the G_a-invariance of S) the following result, reformulated somewhat for our purposes.

THEOREM. *Put* $\eta = (1-w_0)\ldots(1-w_n)$. *The bilinear form* S *over* $J_a\,\eta \cong \tilde{H}_n(V_a)$ *is given by*

$$S(x\eta, y\eta) = f(\bar{y}\, x\eta)\ , \qquad (x,y \in J_a),$$

where $f : J_a \to \mathbb{Z}$ *is the additive homomorphism with*

$$f(1) = -f(w_0 \ldots w_n) = (-1)^{\frac{n(n-1)}{2}}$$

$$f(w) = 0 \ \text{ for } \ w \in G_a\ , \quad w \neq 1, \quad w \neq w_0 \ldots w_n\ ,$$

and where $y \longmapsto \bar{y}$ *is the ring automorphism of the group ring* J_a *induced by* $w \longmapsto w^{-1}$ $(w \in G_a)$.

§ 2. The quadratic form of V_a.

Let G be a finite abelian group, $J(G)$ its group ring. The ring automorphism of $J(G)$ induced by $g \mapsto g^{-1}$ $(g \in G)$ is denoted by $x \mapsto \bar{x}$ $(x \in J(G))$. Give an element $\eta \in J(G)$ and a function $f : G \to \mathbb{Z}$. The additive homomorphism $J(G) \to \mathbb{Z}$ induced by f is also called f. Put $\hat{f} = \sum_{w \in G} f(w)w$. We assume

a) $f(\bar{x}\eta) = f(x\eta)$ for all $x \in J(G)$, [equivalently $\hat{f}\bar{\eta} = \bar{\hat{f}}\eta$]

or

b) $f(\bar{x}\eta) = -f(x\eta)$ for all $x \in J(G)$, [equivalently $\hat{f}\bar{\eta} = -\bar{\hat{f}}\eta$] .

The bilinear form S over the lattice $J(G)\eta$ defined by

$$S(x\eta,\ y\eta) = f(\bar{y}x\eta),\ (x,\ y \in J(G)) ,$$

is symmetric in case a), skew symmetric in case b). Since S is a form with <u>integral</u> coefficients, its determinant is well-defined. The signature

$$\tau(S) = \tau^+(S) - \tau^-(S), \quad \text{case a)} ,$$

is the number $\tau^+(S)$ of positive minus the number $\tau^-(S)$ of negative diagonal entries in a diagonalisation of S over $\mathbb{R}$. Let χ run through the characters of G.

LEMMA. <u>With the preceeding assumptions</u>

$$\pm \det S = \prod_{\chi(\eta) \neq 0} \chi(\hat{f}) \ . \ \text{order of the torsion subgroup of } J(G)/J(G)\eta$$

<u>and in case</u> a)

$$\tau^+(S) = \text{number of characters } \chi \text{ with } \chi(\hat{f}\bar{\eta}) > 0$$

$$\tau^-(S) = \text{number of characters } \chi \text{ with } \chi(\hat{f}\bar{\eta}) < 0 \ .$$

The proof is an exercise as in [1], p. 444.

The lemma and the last theorem of § 1 imply for the affine hypersurface $V_a = V(a_o, \ldots, a_n)$ the

THEOREM. Let S be the intersection form of V_a. Then

$$(1) \qquad \pm \det S = \prod_{1 \le k_j \le a_j - 1} (1 - \varepsilon_0^{k_0} \varepsilon_1^{k_1} \cdots \varepsilon_n^{k_n})$$

where $\varepsilon_j = \exp(2\pi i/a_j)$. For n even, we have

$$(2) \qquad \begin{aligned} \tau^+(S) &= \text{number of } (n+1)\text{-tpls of integers } (x_0, \dots, x_n),\ 0 < x_j < a_j, \\ &\quad \text{with } 0 < \sum_{j=0}^{n} \frac{x_j}{a_j} < 1 \bmod 2\mathbb{Z} \\ \tau^-(S) &= \text{number of } (n+1)\text{-tpls of integers } (x_0 \dots, x_n),\ 0 < x_j < a_j, \\ &\quad \text{with } -1 < \sum_{j=0}^{n} \frac{x_j}{a_j} < 0 \bmod 2\mathbb{Z}. \end{aligned}$$

See [5] for details.

REMARK. The intersection form S of $V(a_0, \dots, a_n)$ with $n \equiv 0 \bmod 2$ is even, i.e. $S(x,x) \equiv 0 \bmod 2$ for $x \in \tilde{H}_n(V_n)$. Therefore, by a well-known theorem, $\det S = \pm 1$ implies $\tau^+(S) - \tau^-(S) = \tau(S) \equiv 0 \bmod 8$.

Following MILNOR we introduce for $a = (a_0, \dots, a_n)$ the graph $\Gamma(a)$: $\Gamma(a)$ has the $(n+1)$ vertices $a_0, \dots, a_n$. Two of them (say a_i, a_j) are joined by an edge if and only if the greatest common divisor (a_i, a_j) is greater than 1. Then we have [5]

LEMMA. $\det S$ as given in the preceeding theorem equals ± 1 if and only if $\Gamma(a)$ satisfies

a) $\Gamma(a)$ has at least two isolated points, or,

b) it has one isolated point and at least one connectedness component K with an odd number of vertices such that $(a_i, a_j) = 2$ for $a_i, a_j \in K$ $(i \neq j)$.

Now suppose n even and $a = (a_0, \dots, a_n) = (p, q, 2, \dots, 2)$ with p, q odd and $(p,q) = 1$. Then $\det S = \pm 1$ and

(3) $$(-1)^{n/2}.\tau(S) = \frac{(p-1)(q-1)}{2} + 2(N_{p,q} + N_{q,p}) ,$$

where $N_{p,q}$ is the number of $q.x$ $(1 \leqq x \leqq \frac{p-1}{2})$ whose remainder mod p of smallest absolute value is negative. This follows from the preceding theorem. Observe that by the above remark $\tau(S)$ is divisible by 4 (even by 8) and that this is related to one of the proofs of the quadratic reciprocity law ([1], p. 450).

In particular, for n even and $(a_0,\dots, a_n) = (3, 6k-1, 2,\dots, 2)$ the signature $\tau(S)$ equals $(-1)^{n/2}.8k$.

§ 3. Exotic spheres.

A k-dimensional compact oriented differentiable manifold is called a k-sphere if it is homeomorphic to the k-dimensional standard sphere. A k-sphere not diffeomorphic to the standard k-sphere is said to be exotic. The first exotic sphere was discovered by MILNOR in 1956. Two k-spheres are called equivalent if there exists an orientation preserving diffeomorphism between them. The equivalence classes of k-spheres constitute for $k \geqq 5$ a finite abelian group Θ_k under the connected sum operation. Θ_k contains the subgroup bP_{k+1} of those k-spheres which bound a parallelizable manifold. bP_{4m} $(m \geqq 2)$ is cyclic of order

$$2^{2m-2}(2^{2m-1} - 1) \text{ numerator } (\frac{4B_m}{m}) ,$$

where B_m is the m-th Bernoulli number. Let g_m be a generator of bP_{4m} . If a $(4m-1)$-sphere Σ bounds a parallelizable manifold B of dimension 4m, then the signature $\tau(B)$ of the intersection form of B is divisible by 8 and

(1) $$\Sigma = + \frac{\tau(B)}{8} g_m$$

(g_m should be chosen in such a way that we have always the plus-sign in (1)). For $m = 2$ and 4 we have

$$bP_8 = \Theta_7 = \mathbb{Z}_{28}\ , \quad bP_{12} = \Theta_{11} = \mathbb{Z}_{992}\ .$$

All these results are due to MILNOR-KERVAIRE. The group bP_{2n} (n odd, $n \geqq 3$) is either 0 or $\mathbb{Z}_2$. It contains only the standard sphere and the KERVAIRE sphere (obtained by plumbing two copies of the tangent bundle of S^n). It is known that bP_{2n} is $\mathbb{Z}_2$ (equivalently that the KERVAIRE sphere is exotic) if $n \equiv 1 \bmod 4$ and $n \geqq 5$ (E. BROWN-F. PETERSON).

Let $V^o_a = V^o(a_o, a_1, \ldots, a_n) \subset \mathbb{C}^{n+1}$ (where $a_j \geqq 2$) be defined by

$$z_o^{a_o} + z_1^{a_1} + \ldots + z_n^{a_n} = 0\ .$$

This affine variety has exactly one singular point, namely the origin of $\mathbb{C}^{n+1}$. Let

$$S^{2n+1} = \{z \mid z \in \mathbb{C}^{n+1}\ ,\ \sum_{j=0}^{n} z_j \bar{z}_j = 1\}\ .$$

Then $\Sigma_a = \Sigma(a_o, \ldots, a_n) = V^o_a \cap S^{2n+1}$ is a compact oriented differentiable manifold (without boundary) of dimension $2n-1$.

THEOREM. <u>Let</u> $n \geqq 3$. <u>Then</u> Σ_a <u>is</u> $(n-2)$-<u>connected</u>. <u>It is a</u> $(2n-1)$-<u>sphere if and only if the graph</u> $\Gamma(a)$ <u>defined in</u> § 2 <u>satisfies the condition</u> a) <u>or</u> b). <u>If</u> Σ_a <u>is a</u> $(2n-1)$-<u>sphere, then it belongs to</u> bP_{2n}. <u>If, moreover,</u> $n = 2m$, <u>then</u>

$$\Sigma_a = \frac{\tau}{8}\, g_m\ ,$$

<u>where</u> $\tau = \tau^+ - \tau^-$ <u>and</u> τ^+, τ^- <u>are as in</u> § 2 (2). <u>In particular</u>

$$\sum_{i=0}^{2m} z_i\, \bar{z}_i = 1$$

$$z_o^3 + z_1^{6k-1} + z_2^2 + \ldots + z_{2m}^2 = 0$$

<u>is a</u> $(4m-1)$-<u>sphere embedded in</u> $S^{4m+1} \subset \mathbb{C}^{2m+1}$ <u>which represents the element</u> $(-1)^m k.g_m \in bP_{4m}$. <u>Example</u> : <u>For</u> $m = 2$ <u>and</u> $k = 1,\dots, 28$ <u>we get the</u> 28 <u>classes of</u> 7-<u>spheres, for</u> $m = 3$ <u>and</u> $k = 1,\dots, 992$ <u>the</u> 992 <u>classes of</u> 11-<u>spheres</u>.

COROLLARY. <u>The affine variety</u> $V^o(a_o,\dots, a_n)$, $n \geqq 3$, <u>is a topological manifold if and only if the graph</u> $\Gamma(a)$ <u>satisfies</u> a) <u>or</u> b) <u>of</u> § 2.

For this theorem and for the case n odd see BRIESKORN [5].

<u>Proof</u>. If we remove from V^o_a the points with $z_n = 0$ we get a space $\tilde{V}_a$ whose fundamental group has $\pi_1(V_a - \{0\}) \cong \pi_1(\Sigma_a)$ as homomorphic image. $\tilde{V}_a$ is fibred over $\mathbb{C}^*$ with $V(a_o,\dots, a_{n-1})$ as fibre which is simply-connected. Thus $\pi_1(\tilde{V}_a) \cong \mathbb{Z}$ and $\pi_1(\Sigma_a)$ is commutative. Because of this and by SMALE-POINCARÉ we have to study only the homology of Σ_a .

Let $V^\varepsilon_a \subset \mathbb{C}^{n+1}$ be the affine variety

$$z_o^{a_o} + z_1^{a_1} + \dots + z_n^{a_n} = \varepsilon$$

$(V_a = V^1_a)$. Let D^{2n+2} be the full ball in $\mathbb{C}^{n+1}$ with center 0 and radius 1 and S^{2n+1}, as before, its boundary. Σ_a is diffeomorphic to $\Sigma^\varepsilon_a = S^{2n+1} \cap V^\varepsilon_a$ for $\varepsilon > 0$ and small. It is the boundary of $B^\varepsilon_a = D^{2n+2} \cap V^\varepsilon_a$ whose interior (for ε small) is diffeomorphic to V^ε_a and V_a . The exact homology sequence of the pair $(B^\varepsilon_a , V^\varepsilon_a)$ shows that Σ_a is $(n-2)$-connected. Using POINCARÉ duality we get the exact sequence

$$0 \to H_n(\Sigma_a) \to H_n(V_a) \xrightarrow{\sigma} \mathrm{Hom}(H_n(V_a),\mathbb{Z}) \to H_{n-1}(\Sigma_a) \to 0$$

where the homomorphism σ is given by the bilinear intersection form S of V_a (see § 2). This determines $H^*(\Sigma_a)$ completely : $H_n(\Sigma_a) = 0$ if and only if $\det S \neq 0$. If $\det S \neq 0$, then $|\det S|$ equals the order of $H_{n-1}(\Sigma_a)$.

The manifold B_a^ε is parallelizable since its normal bundle is trivial. This finishes the proof in view of § 2.

§ 4. Manifolds with actions of the orthogonal group.

$O(n)$ denotes the real orthogonal group with $O(m) \subset O(n)$, $m < n$, by

$$A \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \ (A \in O(m),\ 1=\text{unit of } O(n-m)).$$

Let X be a compact differentiable manifold of dimension $2n-1$ on which $O(n)$ acts differentiably $(n \geqq 2)$. Suppose each isotropy group is conjugate to $O(n-2)$ or $O(n-1)$. Then the orbits are either Stiefel manifolds $O(n)/O(n-2)$ (of dimension $2n-3$) or spheres $O(n)/O(n-1)$ (of dimension $n-1$). Suppose that the 2-dimensional representation of an isotropy group of type $O(n-2)$ normal to the orbit is trivial whereas the n-dimensional representation of an isotropy group of type $O(n-1)$ normal to the orbit is the 1-dimensional trivial representation plus the standard representation of $O(n-1)$. Under these assumptions the orbit space is a compact 2-dimensional manifold X' with boundary, the interior points of X' corresponding to orbits of type $O(n)/O(n-2)$, the boundary points of X' to the orbits of type $O(n)/O(n-1)$. Suppose finally that X' is the 2-dimensional disk.

It follows from the classification theorems of [8] and [9] that the classes of manifolds X with the above properties under equivariant diffeomorphisms are in one-to-one correspondence with the non-negative integers. We let $W^{2n-1}(d)$ be the $(2n-1)$-dimensional $O(n)$-manifold corresponding to the integer $d \geqq 0$. The fixed point set of $O(n-2)$ in $W^{2n-1}(d)$ is a 3-dimensional $O(2)$-manifold, namely $W^3(d)$, which by ([9], § 5, Korollar 6) is the lens space $L(d,1)$. Thus in order to determine the d associated to a given $O(n)$-manifold of our type we just have to look at the integral homology group H_1 of the fixed point set of $O(n-2)$. $W^{2n-1}(0)$ is $S^n \times S^{n-1}$, the manifold $W^{2n-1}(1)$ is S^{2n-1}, the actions of $O(n)$ are easily constructed. Consider for $d \geqq 2$ the manifold $\Sigma(d,2,\ldots,2)$ in $\mathbb{C}^{n+1}$ given by

$$(1) \qquad \begin{aligned} & z_o^d + z_1^2 + \ldots + z_n^2 = 0 \\ & \sum_{i=0}^{n} z_i \bar{z}_i = 1 \end{aligned}$$

(see § 3). It is easy to check that this is an $O(n)$-manifold satisfying all our assumptions. The operation of $A \in O(n)$ on $(z_o, z_1, \ldots, z_n)$ is, of course, given by applying the real orthogonal matrix $A \in O(n)$ on the complex vector $(z_1, \ldots, z_n)$ leaving z_o untouched. The fixed point set of $O(n-2)$ is $\Sigma(d,2,2)$ which is $L(d,1)$, see [6].

THEOREM. *The* $O(n)$-*manifold* $\Sigma(d,2,\ldots,2)$ *given by* (1) *is equivariantly diffeomorphic with* $W^{2n-1}(d)$, $n \geqq 2$. *It can also be obtained by equivariant plumbing of* $d-1$ *copies of the tangent bundle of* S^n *along the graph* A_{d-1}

$$\bullet\!-\!\bullet\!-\!\bullet \cdots \bullet \qquad d-1 \text{ vertices} .$$

For the proof it suffices to establish the $O(n)$-action on the manifold obtained by plumbing and check all properties :
$O(n)$ acts on S^n and on the unit tangent bundle of S^n. Since the action of $O(n)$ on S^n has two fixed points the plumbing can be done equivariantly. The fixed point set of $O(n-2)$ is the manifold obtained by plumbing $d-1$ tangent bundles of S^2 which is well-known to be $L(d,1)$, (see [6], resolution of the singularity of $z_o^d + z_1^2 + z_2^2 = 0$).

The above theorem gives another method to calculate the homology of $\Sigma(d,2,\dots,2)$ and to prove that $\Sigma(d,2,\dots,2)$ for d odd and an odd number of 2's is a sphere. In particular, $\Sigma(3,2,2,2,2,2)$ is the exotic 9-dimensional KERVAIRE sphere (see § 3). The calculation of the ARF invariant of the A_{d-1}-plumbing shows more generally that

$$\Sigma(d,2,\dots,2)\ , \quad (d \text{ odd, an odd number of } 2\text{'s})$$

is the standard sphere for $d \equiv \pm 1 \bmod 8$ and the KERVAIRE sphere for $d \equiv \pm 3 \bmod 8$, in agreement with a more general result in [5].

REMARKS. The $O(n)$-manifold $W^{2n-1}(d)$ coincides with BREDON's manifolds M_k^{2n-1} for $d = 2k+1$, see BREDON [3]. $\Sigma(3,2,2,2)$ is the standard 5-sphere (since $\Theta_5 = 0$). Therefore S^5 admits a differentiable involution α with the lens space $L(3,1)$ as fixed point set and a diffeomorphism β of period 3 with the real projective 3-space as fixed point set. Compare [3]. α and β are defined on $\Sigma(3,2,2,2)$ given by (1) as follows

$$\alpha(z_0,z_1,z_2,z_3) = (z_0,z_1,z_2,-z_3)$$

$$\beta(z_0,z_1,z_2,z_3) = (\varepsilon z_0,z_1,z_2,z_3), \quad \text{where} \quad \varepsilon = \exp(2\pi i/3).$$

Many more such examples of "exotic" involutions etc. which are not differentiably equivalent to orthogonal involutions etc. can be constructed.

§ 5. Manifolds associated to knots.

Let X be a compact differentiable manifold of dimension $2n-1$ on which $O(n-1)$ acts differentiably $(n \geqq 3)$. Suppose each isotropy group is conjugate to $O(n-3)$ or $O(n-2)$ or is $O(n-1)$. Then the orbits are either Stiefel manifolds $O(n-1)/O(n-3)$ (of dimension $2n-5$) or spheres $O(n-1)/O(n-2)$ (of dimension $n-2$) or points (fixed points of the whole action). The representations of the isotropy groups $O(n-3)$, $O(n-2)$ and $O(n-1)$ respectively normal to the orbit are supposed to be the 4-dimensional trivial representation, the 3-dimensional trivial plus the standard representation of $O(n-2)$, the 1-dimensional trivial plus the sum of two copies of the standard representation of $O(n-1)$. The orbit space X' is then a 4-dimensional manifold with boundary. We suppose that X' is the 4-dimensional disk D^4.

Then the points of the interior of D^4 correspond to Stiefel-manifold-orbits, the points of $\partial D^4 = S^3$ to the other orbits. The set F of fixed points corresponds to a 1-dimensional submanifold of S^3, also called F.

We suppose F non-empty and connected, it is then a knot in S^3. We shall call an $O(n-1)$-manifold of dimension $2n-1$ a "knot manifold" if all the above conditions are satisfied.

Let K be the set of isomorphism classes of differentiable knots (i.e. isomorphism classes of pairs (S^3,F) - F a compact connected 1-dimensional submanifold - under diffeomorphisms of S^3). For the following theorem see JÄNICH ([9], § 6), compare also W.C. HSIANG and W.Y. HSIANG [8].

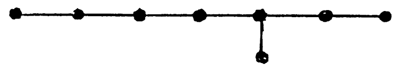
THEOREM. <u>For any</u> $n \geqq 3$ <u>there is a one-to-one correspondence</u>

$$\varkappa_n : K \to \Phi_{2n-1} ,$$

<u>where</u> Φ_{2n-1} <u>is the set of isomorphism classes of</u> $(2n-1)$-<u>dimensional knot manifolds under equivariant diffeomorphisms</u>. $\varkappa_n^{-1}$ <u>associates to a knot manifold the knot</u> F <u>considered above</u>.

REMARK. The 2-fold branched covering of S^3 along a knot F is an $O(1)$-manifold which will be denoted by $\varkappa_2(F)$.

If we plumb 8 copies of the tangent bundles of S^n $(n \geqq 1)$ according to the tree E_8



we get a $(2n-1)$-dimensional manifold $M^{2n-1}(E_8)$. For $n=2$ this is S^3/G, where G is the binary pentagondodecahedral group [6]. For n **odd,** $M^{2n-1}(E_8)$ is the standard sphere, as the ARF invariant shows. For $n = 2m \geqq 4$, the manifold $M^{4m-1}(E_8)$ is an exotic sphere, it is the famous MILNOR sphere which represents the generator $\pm g_m$ of bP_{4m} (see § 3).

$M^{2n-1}(E_8)$ admits an action of $O(n-1)$ as follows : $O(n-1)$ operates as subgroup of $O(n+1)$ on S^n and thus on the unit tangent bundle of S^n. The action on S^n leaves a great circle fixed.

When plumbing the eight copies of the tangent bundle we put the center of the plumbing operation always on this great circle ; (for one copy, corresponding to the central vertex of the E_8-tree, we need three such centers, therefore, we cannot have an action of $O(n)$, which has only 2 fixed points on S^n.) Then the action of $O(n-1)$ on each copy of the tangent bundle is compatible with the plumbing and extends to an action of $O(n-1)$ on $M^{2n-1}(E_8)$ which, for $n \geqq 3$, becomes a knot manifold as can be checked. The resulting knot can be seen on a picture attached at the end of this lecture. The speaker had convinced himself that this is the torus knot $t(3,5)$, but ZIESCHANG and VOGT showed him a better proof. This implies the

THEOREM. Suppose $n \geqq 3$. Then $\varkappa_n(t(3,5))$ is equivariantly diffeomorphic to $M^{2n-1}(E_8)$ with the $O(n-1)$-action defined by equivariant plumbing. (This is still true for $n=2$, see Remark above).

We now consider the manifold $\Sigma(p,q,2,2,\ldots,2) \subset \mathbb{C}^{n+1}$ given by the equations (see § 3)

$$z_o^p + z_1^q + z_2^2 + \ldots + z_n^2 = 0$$

$$\sum_{i=0}^{n} z_i \bar{z}_i = 1 \qquad (n \geqq 3).$$

This is an $O(n-1)$-manifold, the action being defined similarly as in § 4. Suppose $(p,q) = 1$. Then it can be shown that $\Sigma(p,q,2,2,\ldots,2)$ is a knot manifold : It is $\varkappa_n(t(p,q))$ where $t(p,q)$ is the torus knot. Therefore, by the preceeding theorem we have an equivariant diffeomorphism

$$M^{2n-1}(E_8) \cong \Sigma(3,5,\underbrace{2,\ldots,2}_{n-1}) .$$

This gives a different proof (based on the classification of knot manifolds) that $\Sigma(3,5,\underbrace{2,\ldots,2}_{2m-1})$ represents for $m \geqq 2$ a generator of bP_{4m} . (compare § 3).

§ 6. A theorem on knot manifolds.

Let F be a knot in S^3. Then the signature $\tau(F)$ can be defined in the following way which MILNOR explained to the speaker in a letter. MILNOR also considers higher dimensional cases. We cite from his letter, but restrict to classical knots :

Let X be the complement of an open tubular neighbourhood of F in S^3. Then the cohomology

$$H^* = H^*(\hat{X}, \partial\hat{X}; \mathbb{R})$$

where $\hat{X}$ is the infinite cyclic covering of X, satisfies Poincaré duality just as if $\hat{X}$ were a 2-dimensional manifold bounded by F.

In particular the pairing

$$\cup : H^1 \otimes H^1 \to H^2 \simeq \mathbb{R}$$

is non-degenerate. Let t denote a generator for the group of covering transformations of $\hat{X}$. Then for $a,\ b \in H^1$ the pairing

$$< a,b > = a \cup t^*b + b \cup t^*a$$

is symmetric and non-degenerate. Hence, the signature $\tau^+(F) - \tau^-(F) = \tau(F)$ is defined.

There exist earlier definitions of the signature by MURASUGI [13] and TROTTER [17]. The signature is a cobordism invariant of the knot. A cobordism invariant mod 2 was introduced by ROBERTELLO [15] inspired by an earlier paper of KERVAIRE-MILNOR. Let F be a knot and Δ its Alexander polynomial, then the ROBERTELLO invariant $c(F)$ is an integer mod 2, namely

$$c(F) = 0, \quad \text{if} \quad \Delta(-1) \equiv \pm 1 \bmod 8$$

$$c(F) = 1, \quad \text{if} \quad \Delta(-1) \equiv \pm 3 \bmod 8$$

We recall that the first integral homology group of $\varkappa_2(F)$, the 2-fold branched covering of the knot F (see a remark in § 5), is always finite, its order is odd, and equals up to sign the determinant of F. We have $\pm \det F = \Delta(-1)$.

THEOREM. Let F be a knot, then $\varkappa_n(F)$, $n \geqq 2$, is the boundary of a parallelizable manifold. For n odd, $\varkappa_n(F)$ is homeomorphic to S^{2n-1} and thus represents an element of bP_{2n}, it is the standard sphere if

$c(F) = 0$, the KERVAIRE sphere if $c(F) = 1$. If $n = 2m$, then $\varkappa_{2m}(F)$ is $(2m-2)$-connected and $H_{2m-1}(\varkappa_{2m}(F),\mathbb{Z}) \simeq H_1(\varkappa_2(F),\mathbb{Z})$. For $m \geqq 2$ it is homeomorphic to S^{4m-1} if and only if $\det F = \pm 1$. Then $\varkappa_{2m}(F)$ represents (up to sign) an element of bP_{4m} which is $\pm \frac{\tau(F)}{8} \cdot g_m$ (see § 3).

The proof uses an equivariant handlebody construction starting out from a Seifert surface [16] spanned in the knot F. For simplicity, not out of necessity, we have disregarded orientation questions in § 5 and § 6.

REMARK. § 2(3) gives up to sign a formula for the signature of the torus knot $t(p,q)$, (p,q odd with $(p,q) = 1$).

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ERRATUM

Page 314-07. Ligne 4 du bas, au lieu de "Let g_m be a generator of bP_{4m}." lire: "Let g_m be the Milnor generator of bP_{4m}, see p. 314-14."