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THE HILBERT MODULAR GROUP, RESOLUTION OF THE SINGULARITIES AT THE CUSPS AND RELATED PROBLEMS

by F. HIRZEBRUCH

§ 1. The Hilbert modular group and the cusps.

Let k be a real quadratic field over $\mathbb{Q}$ and $\underline{o}$ the ring of algebraic integers in k. Let $x \mapsto x'$ be the non-trivial automorphism of k. The Hilbert modular group

(1) $$SL_2(\underline{o}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \underline{o}, \ ad - bc = 1\}$$

acts on $H \times H$ where H is the upper half plane of $\mathbb{C}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z_1, z_2) = \left(\frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'}\right).$$

The group $G = SL_2(\underline{o})/\{1,-1\}$ acts effectively. For a description of a fundamental domain of G, see Siegel [13].

For any point $x \in H \times H$, the isotropy group $G_x \subset G$ is finite cyclic. The singular points of the complex space $H \times H/G$ correspond bijectively to the finitely many conjugacy classes of maximal finite cyclic subgroups in G. Their number has been determined by Prestel [12] (see also Gundlach [7]). If, for example,

$$D \equiv 1 \ (4), \quad D \not\equiv 0 \ (3), \quad D > 5, \quad D \text{ square free}, \quad k = Q(\sqrt{D}),$$

then there are $h(-D)$ singular points of order 2 and $h(-3D)$ singular points of order 3 where $h(a)$ denotes the ideal class number of $\mathbb{Q}(\sqrt{a})$. (Assume a to be square free.)

G acts on the projective line $k \cup \{\infty\}$ by

$$x \mapsto \frac{ax + b}{cx + d}.$$

There are finitely many orbit classes. The elements of $(k \cup \{\infty\})/G$ are called cusps. They correspond bijectively to the ideal classes of $\underline{o}$. If $x = \frac{m}{n}$ (where

$m , n \in \underline{o}$) belongs to a certain orbit, then (m,n) is a corresponding ideal. We denote by C the group of ideal classes in $\underline{o}$. (The principal ideals represent the unit element of C .) $H \times H/G$ can be compactified by adding finitely many points, namely the cusps. The resulting space

$$\overline{H \times H/G} = (H \times H/G) \cup C$$

is a compact algebraic surface (compare Gundlach [5]) with isolated singularities (the quotient singularities, as explained above, and the finitely many cusps). We wish to resolve the singularities. This is well-known for the quotient singularities (see, for example, [9] § 3.4). Object of this lecture is to do it for the cusps. For this we have to study the neighborhood of a cusp x in $\overline{H \times H/G}$ and the local ring at x .

We sometimes denote a cusp and a representing element $\frac{m}{n}$ $(m , n \in \underline{o})$ by the same symbol x . Let $G_x = \{\gamma \mid \gamma \in G , \gamma x = x\}$. We cannot, in general, transform $x = \frac{m}{n}$ to ∞ by an element of G , but it can be done by a matrix A with coefficients in k . Put $\underline{a} = (m,n)$. Then, following Siegel [13], we take

$$A = \begin{pmatrix} m & u \\ n & v \end{pmatrix} \in SL_2(k)/\{1,-1\} \tag{2}$$

where $u , v \in \underline{a}^{-1}$ (fractional ideal) and define

$$G_x^\infty = A^{-1} G_x A . \tag{3}$$

Then

$$G_x^\infty = \{ \begin{pmatrix} \varepsilon & w \\ 0 & \varepsilon^{-1} \end{pmatrix} \mid w \in \underline{a}^{-2} \}/\{1,-1\} \tag{4}$$

where ε is a unit of k . If we agree to consider a matrix always as a projective transformation, then

$$G_x^\infty = \{\begin{pmatrix} \varepsilon^2 & w \\ 0 & 1 \end{pmatrix} \mid \varepsilon \text{ unit}, w \in \underline{a}^{-2} \} . \tag{5}$$

The group U of positive units of k is infinite cyclic. Let e_o be the generator with $e_o > 1$. It is called the fundamental unit. Let U^+ be the group of totally positive units, i.e.

$$U^+ = \{ \varepsilon \mid \varepsilon \in U , \varepsilon > 0 , \varepsilon' > 0 \} .$$

Equation (5) is a motivation to study data (M,V) where :

(6)
1) M is a $\mathbb{Z}$-module of rank 2 contained in k ;

2) V is a subgroup $\neq \{1\}$ of the group U_M^+ of totally positive units which leave M invariant under multiplication (as is well-known $U_M^+ \neq \{1\}$).

Given the data (6) we have a group

(7) $$\{\begin{pmatrix} \varepsilon & w \\ 0 & 1 \end{pmatrix} \mid \varepsilon \in V\ ,\ w \in M\}\ .$$

In analogy to (4) such groups occur for cusps which are singular points of the compactified orbit spaces F of more general discontinuous groups acting on $H \times H$ (subgroups of finite index of certain finite extensions of G). In (4) we have $M = \underline{a}^{-2}$ and $V = U^2$ and $U_M^+ = U^+$.

Data (M,V) as in (6) determine a torus bundle X over the circle :

(8) $$V \simeq \pi_1(S^1)\ ,\quad (M \otimes_{\mathbb{Z}} \mathbb{R})/M = \text{Torus}$$

$\pi_1(S^1)$ acts on the torus. X is associated to the universal cover of S^1 . The following proposition seems to be well-known. I know it from J.-P. Serre. It follows, for example, from the information given in [5].

PROPOSITION.- <u>If a cusp with data</u> (M,V) <u>is singular point of an algebraic surface</u> F <u>(see above), then its neighborhood boundary is the torus bundle</u> X <u>defined by</u> (8). (For "neighborhood boundary" see, for example, [10].)

The local ring for a cusp (M,V) was described by Gundlach [5]. Let $M^0 \subset \mathbb{R}^2$ be the $\mathbb{Z}$-module of all $x \in \mathbb{R}^2$ such that

(9) $$x_1 w + x_2 w' \in \mathbb{Z} \qquad \text{for all } w \in M\ .$$

M^0 has rank 2 . We have by (9) a bilinear pairing

$$B : M^0 \times M \to \mathbb{Z}\ .$$

V acts on B such that $B(\varepsilon x,w) = B(x,\varepsilon w)$ for $\varepsilon \in V$, $x \in M^0$, $w \in M$.

PROPOSITION.- The local ring for the cusp (M,V) consists of all "convergent" Fourier series

$$f(z_1,z_2) = \sum_{x\in M^0} a_x e^{2\pi i(x_1z_1+x_2z_2)} \tag{10}$$

where $a_x \neq 0$ only if both $x_1 > 0$ and $x_2 > 0$ or $x = 0$, and where $a_{\varepsilon x} = a_x$ for $\varepsilon \in V$. "Convergent" means that f converges for $\mathrm{Im}(z_1).\mathrm{Im}(z_2) > c$ where c is a constant depending on f.

§ 2. Binary indefinite quadratic forms.

Let M be a $\mathbb{Z}$-module of rank 2 contained in k. The function

$$w \mapsto ww' = N(w) \quad (\text{norm of } w) \tag{11}$$

is a quadratic form $M \to \mathbb{Q}$ which is indefinite and does not represent 0. We orient M by the basis (β_1,β_2) of M with $\beta_1\beta_2' - \beta_2\beta_1' > 0$.

We now study oriented $\mathbb{Z}$-modules M of rank 2 and quadratic forms

$$f : M \to \mathbb{Q}$$

which are indefinite and do not represent 0. No specific field k is given.

We call (M_1,f_1) and (M_2,f_2) equivalent if there exists an isomorphism $M_1 \to M_2$ of oriented $\mathbb{Z}$-modules which carries f_1 in tf_2 where t is a positive rational number.

Every (M,f) is equivalent to a quadratic form

$$g : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$$

where $\mathbb{Z}\times\mathbb{Z}$ is canonically oriented and such that for $(u,v) \in \mathbb{Z}\times\mathbb{Z}$

$$g(u,v) = au^2 + buv + cv^2 \tag{12}$$

with $(a,b,c) = 1$. Then $b^2 - 4ac$ is called the discriminant of f. It depends only on the equivalence class of f and is a positive integer which is not a perfect square. The real number

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad \text{where } \sqrt{b^2 - 4ac} > 0 ,$$

is called the <u>first root</u> of g.

We take the unique continued fraction

$$r_1 = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cfrac{1}{a_4 - \ddots}}}$$

where $a_j \in \mathbb{Z}$ and $a_j \geq 2$ for $j > 1$. A continued fraction will be denoted by $[a_1, a_2, a_3, \ldots]$. Since r_1 is a quadratic irrationality its continued fraction is periodic from a certain point on. Let $(b_1, \ldots, b_p)$ be its primitive period $(b_j \geq 2)$. Observe that the period (2) cannot occur because $[2,2,\ldots] = 1$ is rational.

A sequence of integers $(b_1, \ldots, b_p)$ with $b_j \geq 2$ is called a period of length p, two periods are equivalent if they can be obtained from each other by a cyclic permutation. Such an equivalence class is called a cycle. A cycle is primitive if it is not obtainable from another cycle by an "unramified covering" of degree > 1. Cycles are denoted by $((b_1, \ldots, b_p))$. Thus $((2,3))$ is primitive, but $((2,3,2,3))$ is not. $((b_1, \ldots, b_p))^m$ means the m-fold cover of $((b_1, \ldots, b_p))$. For example $((2,5))^3 = ((2,5,2,5,2,5))$.

THEOREM.- <u>The primitive cycle of the first root depends only on the equivalence class of</u> (M,f). <u>If we associate to each</u> (M,f) <u>this primitive cycle, we obtain a bijective map from the set of equivalence classes of quadratic forms</u> (M,f) <u>to the set of all primitive cycles (where</u> $((2))$ <u>is excluded)</u>.

This theorem is a suitable modification of classical results. It is related to Gauss' reduction theory of quadratic forms [3]. The continued fractions had to be modified also, but all relevant theorems in Perron [11] can be taken over.

To simplify notations we shall indicate a cycle by

$$|s_1,t_1|s_2,t_2|s_3,t_3|\dots ,$$

where s_j is the number of two's occuring in the corresponding position and where $t_j \geq 3$. For example,

$$((2,2,2,2,3,3,2,5)) = |4,3|0,3|1,5| .$$

Let k be a real quadratic field over $\mathbb{Q}$ and d its discriminant ; it is the discriminant of the quadratic form (11) defined over the module $\underline{o} \subset k$. If $a > 0$ (square free) and $k = \mathbb{Q}(\sqrt{a})$, then

$$d = 4a \qquad \text{if } a \equiv 2,3 \mod 4$$

$$d = a \qquad \text{if } a \equiv 1 \mod 4 .$$

Let C be as before the group of ideal classes of $\underline{o}$ and C^+ the group of ideal classes with respect to strict equivalence (for which an ideal is equivalent 1 if it is principal with a totally positive generator). We have $|C^+| : |C| = 2$ or 1 depending on whether the fundamental unit e_o is totally positive or not. The order of C is the class number $h(a)$ for $k = \mathbb{Q}(\sqrt{a})$. If the discriminant of k is d , then C^+ is via (11) in one-to-one correspondence with the set of equivalence classes of quadratic forms (M,f) with discriminant d .

Don Zagier (Bonn) has written a computer program which puts out (the finitely many) primitive cycles for a given discriminant. For $d = 257$ the primitive cycles are

a) $|0,3|14,3|0,17|$

b) $|2,3|6,5|0,9|$

c) $|0,5|6,3|2,9|$.

For $d = 4 \cdot 79$ the primitive cycles are

I	$\mid 0,18 \mid 0,9 \mid$
II	$\mid 15,\ 3 \mid 6,3 \mid$
III	$\mid 2,\ 7 \mid 0,3 \mid 0,4 \mid$
IV	$\mid 1,\ 5 \mid 4,3 \mid 0,3 \mid$
V	$\mid 1,\ 3 \mid 0,3 \mid 4,5 \mid$
VI	$\mid 2,\ 4 \mid 0,3 \mid 0,7 \mid$.

For $k = \mathbb{Q}(\sqrt{257})$ the fundamental unit is not totally positive, the class number $h(257)$ equals 3 . For $k = \mathbb{Q}(\sqrt{79})$ the fundamental unit is totally positive, the class number $h(79)$ equals 3 . The order of C^+ is 6 . The 6 quadratic forms for $d = 4 \cdot 79$ are listed by Gauss [3] Art. 187 and numbered from I to VI corresponding to our table above.

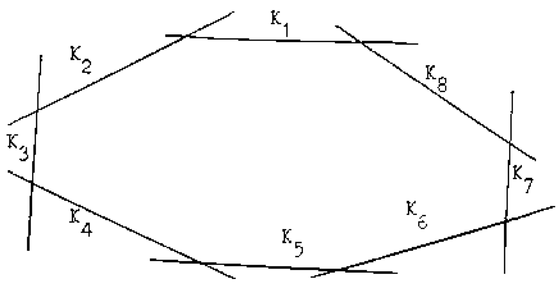
The discriminant $d = 20$, for example, is not the discriminant of a field k . There is one primitive cycle namely $|3,6|$ which belongs to the module $\mathbb{Z}\sqrt{5} \oplus \mathbb{Z}\cdot 1$ contained in $\mathbb{Q}(\sqrt{5})$ and the quadratic form defined on it by (11).

§ 3. Resolution of the cusps.

An isolated singular point x of a complex space of complex dimension 2 admits a resolution by which x is blown up into a system of non-singular curves K_j . For each K_j we have the genus $g(K_j)$ and the self-intersection number $K_j \circ K_j$.

The resolution is minimal (and then uniquely determined) if there is no K_j with $g(K_j) = 0$ and $K_j \circ K_j = -1$. The matrix $(K_i \circ K_j)$ is negative-definite (compare [10]).

The resolution is called <u>cyclic</u> if all $g(K_j)$ are zero (i.e. all curves are rational) and if j can be assumed to run through the residue classes mod q $(q \geq 3)$ such that $K_{j+1} \circ K_j = K_j \circ K_{j+1} = 1$ for all $j \in \mathbb{Z}/q\mathbb{Z}$ (transversal intersection) and $K_r \circ K_s = 0$ for $r - s \not\equiv 0, 1, -1$. Example $(q = 8)$:

(13) 

The following result is a consequence of a theorem in § 4.

THEOREM.- A cusp (M,V) , see (6), admits a cyclic resolution. M determines by (11) and the theorem in § 2 a primitive cycle $c = ((b_1,\ldots,b_p))$. Put $m = [U_M^+ : V]$. Then $q = pm$ and

$$((-K_1 \circ K_1,\ldots,-K_q \circ K_q)) = c^m .$$

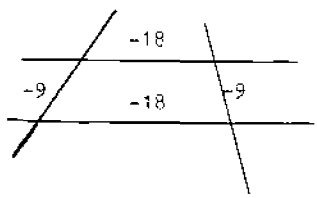
(Exceptional cases $pm = 1$ or 2 . If $c^m = ((b))$ or $((b_1,b_2))$ we have a cycle of 3 curves with self-intersection numbers $-(b+3)$, -2 , -1 or $-(b_1+1)$, -1 , $-(b_2+1)$ respectively.)

The cyclic resolution is the minimal one with these exceptions which can be blown down to minimal ones looking like this :

Examples.- For $k = \mathbb{Q}(\sqrt{a})$ with $a > 1$ (square free) and G as in § 1 we have $h(a)$ cusps ($h(a)$ = order of the ideal class group C , see § 2). Each cusp has the $\mathbb{Z}$-module $\underline{a}^{-2}$ where the ideal $\underline{a}$ represents an element of C . If $\underline{a}$ and $\underline{b}$ give the same element in C , then the $\mathbb{Z}$-modules $\underline{a}^{-2}$ and $\underline{b}^{-2}$ are obtainable from each other by multiplication with a totally positive number and (as fractional ideals) represent the same element of C^+ . Thus we have a homomorphism

$$\rho : C \rightarrow C^+ .$$

The resolution of a cusp $x \in C$ is given by the equivalence class of the quadratic form belonging to $\rho(x)$ or rather by its corresponding primitive cycle c (§ 2). The cycle of the resolution is c^m where $m = 2$ if the fundamental unit e_o of k is totally positive, otherwise $m = 1$. For $k = \mathbb{Q}(\sqrt{79})$ and G as in § 1, we have three cusps. We have to analyze what are the squares in C^+ and their periods. In the list of § 2 the squares are I, IV, V. The cusps IV, V give the same singularity (the periods are just reversed). They go over into each other by the permutation σ of the factors of $H \times H$ (which leaves the cusp I invariant). The resolution of the cusp I looks like :

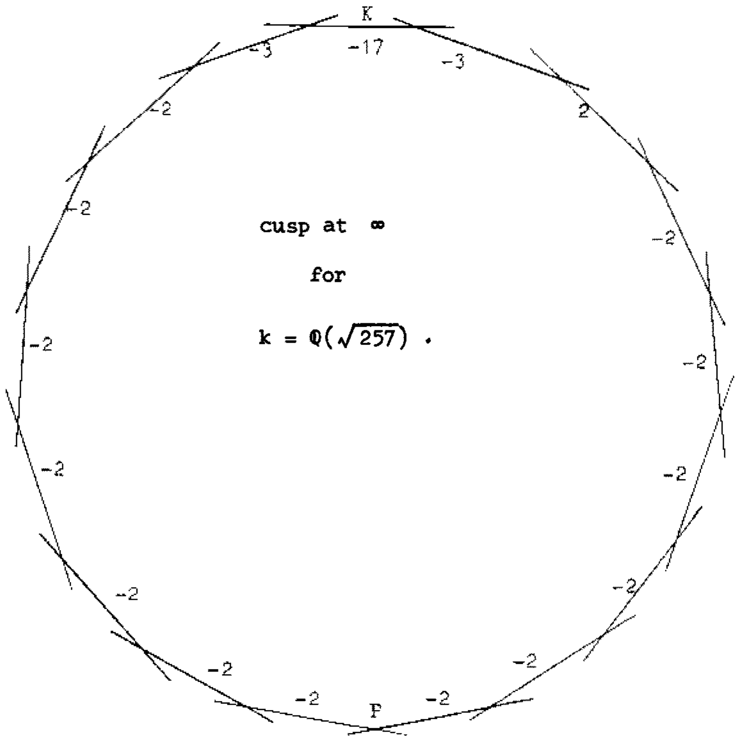


where we have indicated the self-intersection numbers. The (minimal) resolution of IV has 16 curves.

For $k = \mathbb{Q}(\sqrt{257})$ we have $C = C^+$ and $m = 1$. The resolutions of the three cusps are given by the primitive cycles written down in § 2.

The permutation σ on $H \times H$ carries the cusp b) into the cusp c) whereas on the cusp a) it carries the curve K with self-intersection number -17 into itself, has the intersection point P of two curves of self-intersection number -2 as fixed point

(14)



and otherwise interchanges the curves according to the symmetry of the continued fraction of a quadratic irrationality w , which is equivalent to $-w'$ under $SL_2(\mathbb{Z})$ (Theorem of Galois, see [11] § 23). The corresponding singularity of $\overline{(H \times H/G)}/\sigma$ is a quotient singularity admitting a "linear resolution"

(compare [9] § 3.4)

obtained by "dividing" the diagram (14) by σ and using that curves of self-intersection number -1 can be "blown down" .

§ 4. Construction of cyclic singularities.

Let $b_1, b_2, \ldots, b_q$ $(q \geq 3)$ be a sequence of integers ≥ 2 not all equal 2. For $q = 3$ also sequences $(a+3, 2, 1)$ and $(a_1 + 1, 1, a_2 + 1)$ with $a \geq 3$ and $a_1 \geq 3$ or $a_2 \geq 3$ are admitted. Let j run through $\mathbb{Z}/q\mathbb{Z}$. Consider the matrix (c_{rs}), where $r, s \in \mathbb{Z}/q\mathbb{Z}$, with

$$c_{j+1,j} = c_{j,j+1} = 1, \quad c_{jj} = -b_j, \quad c_{rs} = 0 \quad \text{otherwise.}$$

LEMMA.- *Under the preceding assumptions the matrix* (c_{rs}) *is negative-definite.*

Let k run through the integers and define b_k to be equal to b_j above if $k \equiv j \mod q$. We now do a construction as in [9] § 3.4. For each k take a copy R_k of $\mathbb{C}^2$ with coordinates u_k, v_k. We define R'_k to be the complement of the line $u_k = 0$ and R''_k to be the complement of the line $v_k = 0$.

The equations

$$u_k = u_{k-1}^{b_k} v_{k-1}$$

$$v_k = 1/u_{k-1}$$

give a biholomorphic map $\varphi_{k-1} : R'_{k-1} \to R''_k$. If we make in the disjoint union $\cup R_k$ the identifications given by the φ_{k-1} we get a complex manifold Y in which we have a string of compact rational curves S_k non-singularly imbedded. S_k is given by $u_k = 0$ "in the k-th coordinate system" and by $v_{k-1} = 0$ in the $(k-1)$-th coordinate system. S_k, S_{k+1} intersect in just one point transversally. S_i, S_k $(i < k)$ do not intersect, if $k - i \neq 1$. The self-intersection number $S_k \circ S_k$ equals $-b_k$. The complex manifold Y admits a biholomorphic map $T : Y \to Y$ which sends a point with coordinates u_k, v_k in the k-th coordinate system to the point with the same coordinates in the $(k+q)$-th coordinate system, thus $T(S_k) = S_{k+q}$. The main point is the existence of a tubular neighborhood Y^o of $\cup S_k$ on which the infinite cyclic group $Z = \{T^n \mid n \in \mathbb{Z}\}$ operates freely such that Y^o/Z is a complex manifold in which q rational curves $K_1 \cup \ldots \cup K_q = \cup S_k/Z$ are embedded. Their intersection behaviour is given by the negative-definite matrix c_{rs} (see Lemma).

According to Grauert [4] the curves $K_1 \cup \ldots \cup K_q$ can be blown down to a singular point x in a complex space where x has by construction a cyclic resolution as defined in § 3.

THEOREM.- Let $\beta = [b_1,\ldots,b_q, b_1,\ldots,b_q,\ldots]$. Then $M = \mathbb{Z}\beta \oplus \mathbb{Z}\cdot 1$ is a $\mathbb{Z}$-module contained in $k = \mathbb{Q}(\beta)$. Suppose $((b_1,\ldots,b_q))$ is the m-th power of a primitive cycle. Then the local ring at the singular point x constructed above is isomorphic to the local ring described in the second proposition of § 1 provided $[U_M^+ : V] = m$.

The proof will be published elsewhere.

§ 5. Applications.

The resolution of the cusps can be used to calculate certain numerical invariants of $H \times H/G$, $(\overline{H \times H/G})/\sigma$, for example, where $\sigma : H \times H \to H \times H$ is the permutation of the factors as before. We have to use a result of Harder [8]. Compare the lecture of Serre in this Seminar. We mention two cases.

1. For a cusp $x = (M,V)$ with a resolution as in the theorem of § 3 we put

$$\varphi(x) = \frac{1}{3}\Big(\sum_{j=1}^{q} K_j \circ K_j\Big) + q .$$

The number $\varphi(x)$ is essentially the value at 1 of a certain L-function. $\varphi(x)$ vanishes if the quadratic form f on M (see (11)) is equivalent to $-f$ (under an automorphism of M which need not be orientation preserving).

THEOREM.- Suppose $a > 6$, square free, $a \not\equiv 0$ (3). Put $k = \mathbb{Q}(\sqrt{a})$. Using the notation of § 1 we have :

The signature of the (non-compact) rational homology manifold $H \times H/G$ equals $\sum_{x \in C} \varphi(x)$.

2. For a prime $p \equiv 1 \mod 4$ we shall calculate the arithmetic genus $\hat{\chi}_p$ of the non singular model of the compact algebraic surface $(\overline{H \times H/G})/\sigma$ for $k = \mathbb{Q}(\sqrt{p})$. Information on the fixed points (see § 1) is needed. The following result is closely related to theorems of Freitag [2] and Busam [1], see in particular [1] § 7.

THEOREM.- Let p be a prime $\equiv 1 \mod 4$ and $p > 5$. Put $k = \mathbb{Q}(\sqrt{p})$. The arithmetic genus $\hat{\chi}_p$ is given by

$$48\,\hat{\chi}_p = 12\,\zeta_k(-1) + 3h(-p) + 4h(-3p) - p + 8\varepsilon + 12\,\delta + 29$$

where $\varepsilon = 1$ for $p \equiv 1 \mod 3$, $\varepsilon = 0$ for $p \equiv 2 \mod 3$, $\delta = 1$ for $p \equiv 1 \mod 8$, $\delta = 0$ for $p \equiv 5 \mod 8$. (ζ_k is the Zeta-function of the field k.)

For $\zeta_k(-1)$ we have the following formula [14]

$$\zeta_k(-1) = \frac{1}{30} \sum_{\substack{b \text{ odd} \\ 1 \leq b < \sqrt{p}}} \sigma_1\left(\frac{p - b^2}{4}\right),$$

where $\sigma_1(n)$ is the sum of the divisors of n.

By calculations of R. Lundquist, Don Zagier and myself there are exactly 24 primes $\equiv 1 \mod 4$ for which the arithmetic genus equals 1, namely all such primes smaller than the prime 193 and 197, 229, 269, 293, 317. For $p = 5$ the surface $(\overline{H \times H/G})/\sigma$ is rational (Gundlach [6]). Which of the 23 others are rational ?

Final joke : At the end of my dissertation [9] I claim that there are no cycles in a resolution. This is nonsense, as I know for a long time, and as this talk proves, I hope.

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