PONTRJAGIN CLASSES OF RATIONAL HOMOLOGY MANIFOLDS AND THE SIGNATURE OF SOME AFFINE HYPERSURFACES

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Let X be a rational homology manifold in the sense of Thom (Symp. Intern. Top. Alg. 1956, p. 54-67, Universidad de México 1958). Thom defines Pontrjagin classes

$$p_i(X) \in H^{4i}(X, \mathbb{Q}).$$

Instead of defining

$$p(X) = 1 + p_1(X) + p_2(X) + \ldots \in H^*(X, \mathbb{Q})$$

one can define

$$\mathcal{L}(X) = 1 + L_1(X) + L_2(X) + \ldots \in H^*(X, \mathbb{Q})$$

(see Hirzebruch, Topological methods in algebraic geometry, third edition, Springer 1966, §1). The classes $p(X)$ and $\mathcal{L}(X)$ determine each other.

Let G_m be the group of m^{th} roots of unity and

$$G_b = G_{b_0} \times \ldots \times G_{b_n} \quad \text{where} \quad b = (b_0, \ldots, b_n),\ b_k \geq 1.$$

Let G_b act on the complex projective space $P_n(C)$ (homogeneous coordinates $t_0, \ldots, t_n$) as follows:

$$\alpha(t_0, \ldots, t_n) = (\alpha_0 t_0, \ldots, \alpha_n t_n),$$

$$\text{where} \quad \alpha = (\alpha_0, \ldots, \alpha_n) \in G_b.$$

The orbit space $P_n(C)/G_b$ is a rational homology manifold. The map $p : P_n(C) \to P_n(C)/G_b$ induces an isomorphism

$$p^* : H^*\left(P_n(C)/G_b, \mathbb{Q}\right) \to H^*\left(P_n(C), \mathbb{Q}\right).$$

$H^*\left(P_n(C), \mathbb{Q}\right)$ is the truncated polynomial ring $\mathbb{Q}[x]/(x^{n+1})$ where $x \in H^2\left(P_n(C), \mathbb{Z}\right)$ is the Poincaré dual of the hyperplane. Bott (not yet published) has calculated the

Pontrjagin classes of $P_n(C)/G_b$.

(1) $$p^*\mathcal{L}\left(P_n(C)/G_b\right) = \frac{1}{[b_0,\ldots,b_n]} \sum_{0\leq\xi<\pi} \prod_{k=0}^{n} \frac{b_k x}{\tanh b_k(x+i\xi)}$$

where $[b_0, \ldots, b_n]$ is the greatest common divisor of $b_0, \ldots, b_n$. The sum is over all real numbers ξ with $0 \leq \xi < \pi$. Observe, however, that for any natural number $a \geq 1$ the term $\left(\tanh a(x + i\xi)\right)^{-1}$ is a power series in x if $a\xi \not\equiv 0$ mod $\mathbb{Z}\pi$. Therefore, $\prod_{k=0}^{n} \frac{b_k x}{\tanh b_k(x+i\xi)}$ is divisible by x^{n+1}, and thus vanishes in the truncated polynomial ring, if for all k we have $b_k\xi \not\equiv 0$ mod $\mathbb{Z}\pi$. Therefore, the above sum (1) is actually only over the finitely many ξ for which

$$b_k\xi \equiv 0 \bmod \mathbb{Z}\pi$$

for at least one k with $0 \leq k \leq n$.

Let N be a common multiple of $b_0, \ldots, b_n$ and consider hypersurface

$$X^N : t_0^N + \ldots + t_n^N = 0$$

in $P_n(C)$. Then $\alpha(X^N) = X^N$ for $\alpha \in G_b$ and $X^N/G_b \subset P_n(C)/G_b$. This "submanifold" X^N/G_b of $P_n(C)/G_b$ has a normal bundle ν in the sense of Thom (loc. cit.). It is a U(1)-bundle whose lift to X^N is the normal bundle of X^N in $P_n(C)$. Observe that for each $\alpha \in G_b$ the set $P_n(C)^\alpha$ of fixed points is transversal to X^N. We obtain for the signature of X^N/G_b

$$\text{sign } X^N/G_b = \text{coefficient of } x^n \text{ in}$$

$$\tanh (Nx) . \sum_{0\leq\xi<\pi} \prod_{k=0}^{n} \frac{x}{\tanh b_k(x+i\xi)} .$$

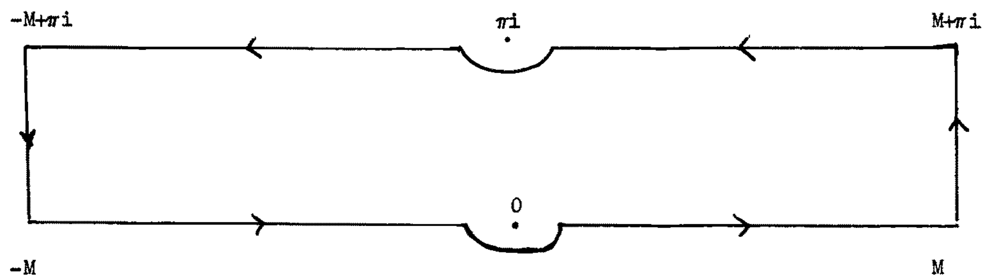
Here we used the fact that the map p has degree $b_0 \ldots b_n/[b_0, \ldots, b_n]$. Since N is a multiple of all the b_k, easy shifting of coordinates yields

(2) $$\text{sign } X^N/G_b =$$

$$\sum_{0\leq\xi<\pi} \text{res}_{i\xi} \left[\tanh Nz \cdot \prod_{k=0}^{n} \frac{1}{\tanh b_k z}\right]$$

<u>where the sum is over those</u> ξ <u>with</u> $0\leq\xi<\pi$ <u>such that</u> $\left(\prod_{k=0}^{n} \tanh b_k z\right)^{-1}$ <u>has a pole in</u> $i\xi$.

Let us denote the expression between brackets in (2) by $g(z)$. We integrate $g(z)$ along the following path γ_M in the z-plane.



The integrals along the "horizontal" parts of γ_M cancel each other since $g(z + \pi i) = g(z)$. The sum of the integrals along the vertical parts of γ_M converges for $M \to \infty$ to

$$\pi i\left(1 - (-1)^n\right),$$

because $\tanh(M + iy)$ converges uniformly to 1 or -1 respectively if M converges to $+\infty$ or $-\infty$ respectively. The real dimension of X^N/G_b is $2(n - 1)$. Therefore, we suppose n odd from now on. Otherwise the signature vanishes by definition. We obtain from (2)

$$\text{(3)} \qquad \operatorname{sign} X^N/G_b = 1 - \sum_{0 \leq \eta < \pi} \operatorname{res}_{i\eta} g(z)$$

where the sum is now over those η with $0 \leq \eta < \pi$ such that $\tanh Nz$ has a pole in $i\eta$, which means

$$\eta = \frac{\pi j}{2N}, \quad j \text{ odd}, \quad 1 \leq j < 2N.$$

The function $\tanh Nz$ has poles of order 1 at the $i\eta$. The residue in all these poles is $\frac{1}{N}$. By (3) we get (with $a_k = N/b_k$)

$$\text{(4)} \qquad \operatorname{sign} X^N/G_b - 1 =$$

$$\frac{(-1)}{N}^{(n-1)/2} \sum_{\substack{j \text{ odd} \\ 1 \leq j < 2N}} \cot \frac{\pi j}{2a_0} \cot \frac{\pi j}{2a_1} \ldots \cot \frac{\pi j}{2a_n} .$$

Brieskorn (Inventiones math. 2, 1-14 (1966)) has studied the non-singular affine hypersurface $V_{a_1,a_2,\ldots,a_n}$ in C^n given by

(5) $$z_1^{a_1} + z_2^{a_2} + \ldots + z_n^{a_n} + 1 = 0;$$

its signature is related to the theory of exotic spheres.

Let E be the hyperplane $t_0 = 0$ in $P_n(C)$. Let N be any common multiple of $a_0, a_1, \ldots, a_n$ and

(6) $$b_k = \frac{N}{a_k} \quad (\text{for } k = 1, \ldots, n), \quad b_0 = 1, \; a_0 = N.$$

$Y^N = X^N - X^N \cap E$ is given in $P_n(C) - E$ by

$$t_1^N + t_2^N + \ldots + t_n^N + 1 = 0 \quad (\text{put } t_0 = 1).$$

By the map $z_k = t_k^{b_k}$ from $P_n(C) - E$ to C^n $(k = 1, \ldots, n)$ we have

(7) $$V_{a_1,\ldots,a_n} = Y^N/G_b$$

where $G_b = G_{b_0} \times G_{b_1} \times \ldots \times G_{b_n}$ and $b = (1, b_1, \ldots, b_n)$ as in (6).

The Lefschetz theorem on hyperplane sections implies the following fact. $X^N \cap E$ has a tubular neighbourhood T in X^N invariant under G_b. The middle dimensional homology group of T is infinite cyclic with a generator (invariant under G_b) of self-intersection number + 1. By the Novikov additivity of the signature we get from (4) and (7)

Theorem Let n be odd and N any common multiple of $a_1, \ldots, a_n$. Then the signature of the Brieskorn variety $V_{a_1,\ldots,a_n}$ is given by the formula of Zagier

$$(8) \qquad \operatorname{sign} V_{a_1,\ldots,a_n} = \frac{(-1)^{(n-1)/2}}{N} \sum_{\substack{j \text{ odd} \\ 1 \leq j < 2N}} \cot\frac{\pi j}{2N} \cot\frac{\pi j}{2a_1} \cdots \cot\frac{\pi j}{2a_n} .$$

Brieskorn (loc. cit.) gives the following formula

$$\operatorname{sign} V_{a_1,\ldots,a_n} =$$

$$(9) \qquad \# \left\{ 0 < x_k < a_k \;\middle|\; 0 < \sum_{k=1}^{n} \frac{x_k}{a_k} < 1 \mod 2 \right\}$$

$$- \# \left\{ 0 < x_k < a_k \;\middle|\; 1 < \sum_{k=1}^{n} \frac{x_k}{a_k} < 2 \mod 2 \right\} ,$$

[$(x_1, \ldots, x_n)$ are n-tuples of integers].

Zagier has proved by Fourier series and by other methods that the two expressions in (8) and (9) equal each other. The interesting formula (8) is due to him. In virtue of Zagier's result, we have given a new method (involving Pontrjagin classes) to calculate sign $V_{a_1,\ldots,a_n}$ and to prove (9).

We can identify (8) and (9) in the following way (which is essentially Zagier's method). Put

$$((x)) = x - [x] - \tfrac{1}{2} , \quad \text{if } x \text{ is not an integer} ;$$

$$((x)) = 0 , \quad \text{if } x \text{ is an integer.}$$

Then the expression in (9) is

$$(10) \qquad \operatorname{sign} V_{a_1,\ldots,a_n} =$$

$$2 \sum_{0<x_k<a_k} \left(\left(\frac{x_1}{2a_1} + \cdots + \frac{x_n}{2a_n} + \tfrac{1}{2}\right)\right) - \left(\left(\frac{x_1}{2a_1} + \cdots + \frac{x_n}{2a_n}\right)\right) .$$

If $r = \frac{p}{q}$ is any positive rational number (where p, q are natural numbers, not necessarily coprime), then

$$((r)) = \frac{i}{2q} \sum_{j=1}^{q-1} \cot \frac{\pi j}{q} \; e^{2\pi i . jr} \quad .$$

This is a formula of Eisenstein (see Rademacher, Lectures on Analytic Number Theory, Notes, Tata Institute, Bombay 1954-55, p.276). Feeding it into (10) gives (8).

Remark: The Dedekind sums studied by Rademacher are in close relation to formula (8) and the Atiyah-Bott-Singer fixed point theorem applied to the "signature operator" as will be explained elsewhere.

H. A. Hamm (Dissertation Bonn-Göttingen : see also the following paper) has studied the following affine varieties (given by r equations in C^n)

$$c_{j1} z_1^{a_1} + \ldots + c_{jn} z_n^{a_n} + c_{j,n+1} = 0,$$

$j = 1, \ldots, r$ and $r \leqslant n$.
If all $s \times s$ subdeterminants of the $r \times (n+1)$-matrix (c_{jk}) are different from 0 for $1 \leqslant s \leqslant r$, then the affine variety is a non-singular complete intersection of hypersurfaces. (Our conditions are stronger than those of Hamm.) We denote such a variety by $V^r_{a_1,\ldots,a_n}$. Its complex dimension is $n - r$. If we assume $n - r$ to be even, then the same method as above yields

(11) $$\operatorname{sign} V^r_{a_1,\ldots,a_n} =$$

$$- \sum_{\substack{1 \leqslant j < 2N \\ j \text{ odd}}} \operatorname{res}_{\pi i j/2N} \left((\tanh Nz)^r \coth z \prod_{k=1}^{n} \coth \frac{Nz}{a_k} \right)$$

where N is any common multiple of $a_1, \ldots, a_n$. But it seems harder to get a formula similar to (8) or (9) because we have poles of order > 1. Bott's proof of (1) involves also residue calculations and there is in fact a short cut to (8) or (9) from a point on the way to (1). But it seemed amusing to adopt the view of somebody knowing (1) and not its proof and to begin to calculate.