The Canonical Map for Certain Hilbert Modular Surfaces

F. Hirzebruch*

It was a great pleasure for me to participate in the symposium in honor of Shiing-shen Chern. In my lecture I intended to give a survey on Hilbert modular surfaces. But actually I discussed examples of such Hilbert modular surfaces for which specific information is available on their structure as algebraic surfaces. The paper presented here is an extended version of the talk.

Algebraic surfaces are often investigated by means of their pluricanonical maps (see for example Bombieri [2]). The properties of the canonical map itself (given by the sections of the canonical bundle or in the case of Hilbert modular surfaces by the cusp forms of weight 2) are relatively complicated (compare Beauville [1]). In a certain range, namely for minimal surfaces of general type with $2p_g - 4 \leqslant K^2 \leqslant 2p_g - 2$, Horikawa's results are available [21–25]. To apply them, one has to prove that the surface being studied is minimal. For Hilbert modular surfaces this is a difficult problem, which was attacked first by van der Geer and Van de Ven [10]. Van der Geer has obtained many results on the structure of special Hilbert modular surfaces [8, 9] including some of the surfaces studied here.

The rough classification of Hilbert modular surfaces according to rational, $K3$, elliptic, and general type was considered by Hirzebruch, Van de Ven, and Zagier [17, 19, 16]. The present paper tries to show that in some cases a finer classification of the surfaces of general type can be obtained.

1. Some Examples of Canonical Maps

Let X be a nonsingular n-dimensional compact algebraic manifold and $H^0(X, \Omega^n)$ the complex vector space of holomorphic n-forms. An element $\omega \in H^0(X, \Omega^n)$ can be written with respect to a local coordinate system in the form

$$\omega = a(u_1, u_2, \ldots, u_n)\, du_1\, du_2 \cdots du_n,$$

where a is holomorphic. The dimension of $H^0(X, \Omega^n)$ is the geometric genus p_g. We shall sometimes write g instead of p_g. If $\omega_1, \omega_2, \ldots, \omega_g$ is a base of

*Mathematisches Institut der Universität, Wegelstrasse 10, 53 Bonn, Federal Republic of Germany.

$H^0(X, \Omega^n)$, then we have the canonical "map"

$$\iota_K : X \to P_{g-1}(C),$$
$$p \in X \mapsto \omega_1(p) : \omega_2(p): \cdots :\omega_g(p) \in P_{g-1}(C),$$

which is not necessarily everywhere defined.

The vector space $H^0(X, \Omega^n)$ is the space of holomorphic sections of the canonical bundle $\mathcal{K}$ of X. The complete linear system $|\mathcal{K}|$ consists of all canonical divisors (i.e. divisors of elements $\omega \in H^0(X, \Omega^n)$). These are exactly the inverse images under ι_K of the hyperplanes of $P_{g-1}(C)$. (*Canonical divisors are always assumed to be nonnegative if nothing is mentioned to the contrary*.)

If $n = 1$, then X is a compact Riemann surface (algebraic curve) of genus g. If $g = 0$, then ι_K is not defined. For $g = 1$ the image $\iota_K(X)$ is a point. For $g = 2$, the map ι_K realizes X as double cover of $P_1(C)$ with 6 ramification points. For $g = 3$ the following holds (see Griffiths and Harris [12, p. 247]): The canonical map ι_K is a biholomorphic map of X onto a nonsingular curve of degree $2g - 2$ in $P_{g-1}(C)$ (generic case), or the curve X is hyperelliptic and ι_K realizes X as a double cover of the rational normal curve of degree $g - 1$ in $P_{g-1}(C)$ with $2g + 2$ ramification points. The normal curve mentioned is the image of $P_1(C)$ under the map given by the homogeneous polynomials of degree $g - 1$.

For $n = 2$ not much is known about the canonical map. For algebraic surfaces ($n = 2$) recent investigations are due to Beauville [1], but for specific surfaces (e.g. Hilbert modular surfaces) it is difficult to obtain information on ι_K. We now restrict to the case $n = 2$. Let us first recall some basic facts on algebraic surfaces. By $\mathcal{K}$ we denote the canonical bundle, and by K a canonical divisor. For a nonsingular curve S on X we have the adjunction formula

$$KS + SS = -e(S), \tag{1}$$

where KS and SS are intersection numbers and $e(S)$ is the Euler number of S, which equals $2 - 2g(S)$ if S is irreducible. The formula (1) is true also if K is negative.

The irreducible nonsingular curve S is called exceptional (of the first kind) if $g(S) = 0$ and $SS = -1$ (or equivalently $g(S) = 0$ and $KS = -1$). The surface X is called minimal if it does not contain any exceptional curves.

An exceptional curve S is contained in every (nonnegative) canonical divisor (because $KS = -1$). Therefore ι_K is not defined on an exceptional curve. The exceptional curve S on X can be blown down to a point, the resulting surface Y being nonsingular again. The vector spaces $H^0(X, \Omega_X^n)$ and $H^0(Y, \Omega_Y^n)$ are isomorphic. Let $\pi : X \to Y$ be the natural map; then every canonical divisor on X is of the form $\pi^* K_Y + S$, where K_Y is a canonical divisor on Y. For a nonsingular rational curve S with $SS = -2$ we have $KS = 0$; therefore for every canonical divisor K, the curve S either is contained in K or does not meet K. If S is not contained in all canonical divisors, then ι_K maps S to a point.

Let us now assume that the nonsingular irreducible surface X is of *general type*. Then X contains finitely many exceptional curves mutually disjoint. They can be blown down. The resulting surface again may contain exceptional curves. They are mutually disjoint and can be blown down. After a finite number of such blowing-down processes we reach a minimal algebraic surface, the unique (nonsingular) minimal model in the birational equivalence class of X (see for example Griffiths and Harris [12, pp. 510, 573], Bombieri [2], and Hirzebruch and Van de Ven [17]).

The self-intersection number $K^2 = KK$ is an important invariant. Since the characteristic class of the canonical bundle equals $-c_1$, where $c_1 \in H^2(X, Z)$ is the first Chern class of X, we have

$$K^2 = c_1^2[X].$$

Let $e(X)$ denote the Euler number. M. Noether's formula gives the relation

$$e(X) + K^2 = 12\chi(X), \tag{2}$$

where

$$\chi(X) = 1 - q + p_g$$

(q = irregularity = half first Betti number of X) is the arithmetic genus of X (in the terminology of [14]). If an exceptional curve is blown down, then $e(X)$ decreases by one and K^2 increases by one, whereas $\chi(X)$ remains invariant. It is a birational invariant; in fact q and p_g are birational invariants.

For a minimal algebraic surface X of general type, K^2 and $\chi(X)$ are positive. The number K^2 of X is the maximal K^2 of all nonsingular surfaces in the birational equivalence class of X. By (1), the number $K^2 + 1$ equals the genus of a nonsingular irreducible curve C if C is a canonical divisor. We have the inequality due to M. Noether (compare Bombieri [2, p. 208]),

$$K^2 \geqslant 2p_g - 4, \tag{3}$$

and also the Bogomolov–Miyaoka inequality (Miyaoka [28])

$$K^2 \leqslant 3e(X),$$

for which, of course, minimality is not needed.

We shall now give a few classical examples of minimal algebraic surfaces of general type where the canonical map ι_K is well known from the nature of the example.

EXAMPLE 1. Let X be the double cover of $P_2(C)$ ramified along a nonsingular curve of degree 8. We have $p_g = 3$ and $K^2 = 2$. The natural map $X \to P_2(C)$ is the canonical map ι_K. The complete linear system $|\mathcal{K}|$ consists of the lines in $P_2(C)$ lifted to X. Observe that a (negative) canonical divisor of $P_2(C)$ is given by $-3L$ where L is a line and

$$-3L + \tfrac{1}{2} \cdot 8L = L.$$

The surface X is simply connected. The Euler number equals 46 by Noether's formula (2).

EXAMPLE 2. Consider two nonsingular quartic curves A and B in $P_2(C)$ intersecting transversally. Let X be the fourfold cover of the plane obtained by first taking the double cover Y of the plane ramified along A and then the double cover of Y ramified along the lift of B to Y. This construction is actually symmetric in A and B. The surface X admits an action of $Z/2 \times Z/2$ with the plane as orbit space. We may denote the nontrivial elements of $Z/2 \times Z/2$ by α, β, γ in such a way that X/α and X/β are the double planes ramified along A and B respectively, whereas X/γ is the double cover of the plane ramified along $A \cup B$ (with 16 ordinary rational double points as singularities). The surfaces X/α and X/β are rational (Euler number = 10, $K^2 = 2$; they are isomorphic to a plane with 7 points blown up). Each of them contains 56 exceptional curves coming in pairs which are the lifts of the 28 double tangents of A or B respectively. The surfaces X/α and X/β are double covers of the plane by their anticanonical maps (i.e., the lifts of the lines of the plane are exactly the elements of $|\mathcal{K}^{-1}|$). The branching locus of X over X/α (the lift of B to X/α) is a fourfold anticanonical divisor of X/α. The lifts of the anticanonical divisor of X/α to X are the canonical divisors of X. Observe $K + \frac{1}{2}(-4K) = -K$ (on X/α). The map from X to $P_2(C)$ (degree 4) is the canonical map. For X we have $p_g = 3$ and $K^2 = 4$. It is simply connected and has Euler number 44. The elements α, β, γ operate on $H^0(X, \Omega^2)$ by multiplication with $-1, -1, 1$. Thus $H^0(X, \Omega^2)$ can be identified with the space of holomorphic 2-forms on the nonsingular model X_γ of X/γ (obtained by blowing up each of the 16 singularities in a nonsingular rational curve of self-intersection number -2). The natural map of X_γ to $P_2(C)$ is the canonical map (in fact X_γ belongs to the family of surfaces in Example 1; see later remarks).

Example 3. Consider a nonsingular quadric Q in $P_3(C)$. The quadric is isomorphic to $P_1(C) \times P_1(C)$ by the two systems of lines on Q. Let X be the double cover of Q ramified along a nonsingular curve of bidegree (6, 6). We have $p_g = 4$ and $K^2 = 4$. The natural map of X onto Q followed by the embedding of Q in $P_3(C)$ is the canonical map ι_K. The complete linear system $|\mathcal{K}|$ consists of the planes in $P_3(C)$ intersected with Q and lifted to X. The planes intersected with Q are exactly the curves of bidegree (1, 1). Observe that a (negative) canonical divisor on Q is given by $-2L_1 - 2L_2$, where L_1, L_2 are lines on Q in different systems and

$$-2L_1 - 2L_2 + \tfrac{1}{2}(6L_1 + 6L_2) = L_1 + L_2.$$

The surface X is simply connected. The Euler number equals 56.

Example 4. Let X be a nonsingular quintic surface in $P_3(C)$. We have $p_g = 4$ and $K^2 = 5$. The embedding of X in $P_3(C)$ is the canonical map ι_K. The complete linear system $|\mathcal{K}|$ consists of the planes in $P_3(C)$ intersected with X. The surface X is simply connected. The Euler number equals 55.

Example 5. Consider a nonsingular cubic surface W in $P_3(C)$. For such a surface the complete linear system $|\mathcal{K}^{-1}|$ consists of all hyperplane sections. (These hyperplane sections are the nonnegative anticanonical divisors.) A nonnegative divisor on W is anticanonical if and only if it has intersection number 1 with each of the 27 lines on W. The 27 lines are exactly the exceptional curves on W. Let C be a nonsingular curve on W with $C \in |\mathcal{K}^{-4}|$, i.e., C has intersection number 4 with each of the 27 lines. If we realize W as $P_2(C)$ with six points $p_1, \ldots, p_6$ blown up (Griffiths and Harris [12, p. 489]), then C corresponds to a curve of degree 12 in the plane with $p_1, \ldots, p_6$ as quadruple points and no other singularities. Let X be the double cover of W ramified along C; then the complete linear system $|\mathcal{K}|$ of X consists of the *nonnegative anticanonical divisors of W lifted to X.* Observe $K + \frac{1}{2}(-4K) = -K$ (on W). Thus $|\mathcal{K}|$ consists of all lifted hyperplane sections of W. The canonical map of X is the map onto W followed by the embedding of W in the projective space $P_3(C)$. For X we have $p_g = 4$ and $K^2 = 6$. It is simply connected and has Euler number 54.

EXAMPLE 6. Let X be the double cover of $P_2(C)$ ramified along a nonsingular curve of degree 10. We have $p_g = 6$ and $K^2 = 8$. The natural map $X \to P_2(C)$ followed by the Veronese embedding of $P_2(C)$ in $P_5(C)$ is the canonical map. The canonical divisors of X are the quadrics in $P_2(C)$ (i.e. the hyperplane sections of the Veronese surface) lifted to X. The surface X is simply connected and has Euler number 76.

Remark. Examples 2 and 5 can be regarded as special cases of the following construction. A del Pezzo surface (see Manin [27]) of degree $g-1$ in $P_{g-1}(C)$ is obtained as follows ($4 \leqslant g \leqslant 10$). In the plane $P_2(C)$ we blow up 10-g points. The dimension of the space of sections of the anticanonical bundle of this surface is g. The complete linear system $|\mathcal{K}^{-1}|$ consists of all cubics of $P_2(C)$ passing through the 10-g points. The image of $P_2(C)$ under the anticanonical map is a del Pezzo surface W in $P_{g-1}(C)$. In W we take a nonsingular curve representing a fourfold anticanonical divisor and the double cover of W ramified along this curve. This is an algebraic surface X with geometric genus g and $K^2 = 2g - 2$. The canonical map for X is the map of degree 2 to the del Pezzo surface W followed by the embedding of W in $P_{g-1}(C)$. If $g = 3$, the del Pezzo surface can still be introduced, but its anticanonical map realizes it as double cover of the plane; therefore the canonical map of X is of degree 4 (Example 2).

In Examples 1–6 certain "degenerations" may be admitted. In Examples 3, 4, 5 we admit that the quadric, quintic, or cubic surface has rational double points (sometimes called Kleinian singularities; see Brieskorn [3, 5]). These are the singularities which resolve minimally into a configuration of type A_k, D_k, E_6, E_7, E_8 of nonsingular rational curves of self-intersection number -2. For a quadric we can have only one singularity A_1 (quadric cone). Some examples of quintics with rational double points occur in van der Geer and Zagier [11]. For cubics the complete list of possibilities is given by Schäfli [29]; see also Griffiths and Harris [12, p. 640]. A report on singular cubics was given recently by Bruce and Wall [6]. They list the possible combinations of singularities as A_1, $2A_1$, A_2, $3A_1$, A_1A_2, A_3, $4A_1$, $A_2 2A_1$, A_3A_1, $2A_2$, A_4, D_4, $A_3 2A_1$, $2A_2A_1$, A_4A_1, A_5, D_5, $3A_2$, A_5A_1, E_6.

For the ramification curve on the desingularized quadric (cubic) surface in Examples 3 and 5 we require that it represent a threefold (fourfold) anticanonical divisor. The ramification curve in Examples 1, 3, 5, 6 may have singularities, but they are restricted to the condition that the double cover acquires only rational double points. The admissible curve singularities are, with respect to suitable local coordinates,

$$\begin{aligned} x^2 + y^{k+1} &= 0 \qquad (a_k), \\ x(y^2 + x^{k-2}) &= 0 \qquad (d_k;\quad k > 4), \\ x^3 + y^4 &= 0 \qquad (e_6), \\ x(x^2 + y^3) &= 0 \qquad (e_7), \\ x^3 + y^5 &= 0 \qquad (e_8). \end{aligned}$$

The double cover has singularities of type A_k, D_k, E_6, E_7, E_8 respectively. They are resolved to give our modified examples of algebraic surfaces and canonical maps. In Example 2 we admit that A has singularities a_k, d_k, e_6, e_7, e_8, and we then take on the desingularized model of the double cover branched along A a ramification curve B which represents a fourfold anticanonical divisor. Also B is required to have only singularities a_k, d_k, e_6, e_7, e_8.

By Brieskorn's theory [3–5], we know that the algebraic surfaces thus obtained still belong to the same family (up to deformation). They are, in particular, of the same diffeomorphism type.

Minimal algebraic surfaces with $p_g = 3$ and $K^2 = 2$ are called Moishezon surfaces. They are all of the type studied in Example 1. The surfaces of Examples 1, 3, 6 satisfy $K^2 = 2p_g - 4$, i.e., K^2 is for given p_g as small as possible (by Noether's inequality (3)). For $K^2 = 4$ such a surface belongs to Example 3 by a result of Horikawa [22], who has classified surfaces with $K^2 = 2p_g - 4$. For $K^2 = 8$ our Example 6 is only one of several possibilities. In Example 4 we have $K^2 = 2p_g - 3$. Surfaces satisfying this relation are studied by Horikawa in [21] and [23]. In Examples 2 and 5 we have $K^2 = 2p_g - 2$ with $p_g = 3$ or 4 respectively. For these surfaces see [24] and [25]. In [24] Horikawa considers the case $p_g = 4$. In Section 2 of [25] we find the case $p_g = 3$.

2. Minimality Criterion

How to decide whether a given algebraic surface is minimal? Essentially the following criterion was used by Hirzebruch and Van de Ven [18].

Proposition. *Let X be a nonsingular algebraic surface with $K^2 > 0$, on which there exists a nonnegative divisor D with*

$$D^2 = KD = K^2. \tag{1}$$

Then the homology class of $D - K$ is a torsion class. Every exceptional curve of X is contained in D.

Proof. Since $K(D - K) = (D - K)^2 = 0$ and $K^2 > 0$, we have by the Hodge index theorem for divisors (Griffiths and Harris [12, p. 472]) that $D - K$ is homologous to zero mod torsion. For an exceptional curve S we have $DS = KS = -1$; therefore S is contained in D. □

If a nonnegative divisor D on the surface X satisfies (1), if $K^2 > 0$, and if no exceptional curve is contained in D, then X is minimal.

If a nonnegative divisor D on a simply connected surface satisfies (1), then D is a canonical divisor. It is often difficult to give a nonnegative canonical divisor explicitly, just as it is difficult to prove minimality. One could think that it would be easiest to get one's hands on a nonsingular irreducible curve C with $C^2 = KC = K^2$, which then would have genus $K^2 + 1$. A generic canonical divisor on a minimal surface of general type is such a curve C. However, one often does not succeed in finding such a C; rather one gets complicated configurations D of curves with $D^2 = KD = K^2$. In a way, D is a degenerate curve of genus $K^2 + 1$.

The proposition can be generalized. Let m be a positive integer. Instead of (1) we assume $DK = mK^2$, $D^2 = m^2K^2$. Then the homology class of $D - mK$ is a torsion class, and every exceptional curve of X is contained in D.

We shall now indicate configurations of curves which lead to a divisor D satisfying (1).

Assume that we have in Example 1 of Section 1 a line L in the plane which passes through three double points of types $a_{k_1}, a_{k_2}, a_{k_3}$ of the ramification curve C of degree 8 and intersects C in two points transversally. Then the lift of L in the Moishezon surface is a rational curve $\tilde{L}$ with $K\tilde{L} = 2$ together with a configuration of nonsingular rational curves of self-intersection -2 arising from the resolution of the singularities of types $A_{k_1}, A_{k_2}, A_{k_3}$. The result looks like Figure 1. All curves are nonsingular and rational. All intersections are transversal. We have $\tilde{L}\tilde{L} = -4$. There are three chains of (-2)-curves of lengths k_1, k_2, k_3. This is a configuration D satisfying (1) with $K^2 = 2$ (each component S has multiplicity $m_S = 1$ in D).

To check (1) in this case and all further cases we look at each component S of the configuration $D = \sum m_S S$ and prove

$$DS = KS,$$

$$\sum_S m_S KS = K^2.$$

The number KS is known from the adjunction formula if the genus and self-intersection of S are given, whereas DS can be read off from the configuration.

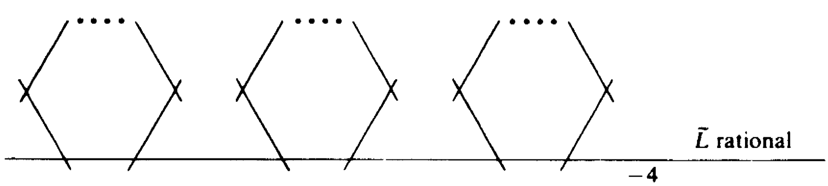

Figure 1. Configuration (I′). $D^2 = KD = 2$.

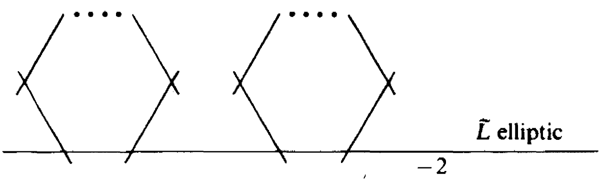

Figure 2. Configuration (I″). $D^2 = KD = 2$.

A second configuration D satisfying (1) with $K^2 = 2$ is the following. Assume that we have in Example 1 a line L in the plane which passes through two double points of types a_{k_1} a_{k_2} of the ramification curve C of degree 8 and intersects C in four points transversally. Then the lift of L in the Moishezon surface is an elliptic curve $\tilde{L}$ with $K\tilde{L} = 2$ together with two chains of (-2)-curves. The result looks like Figure 2 (all multiplicities 1). In [18] the configuration (I″) and two other configurations were used.

Consider the configuration D in Figure 3 of nonsingular rational curves (four (-3)-curves and twelve (-2)-curves). The four (-3)-curves are joined by (-2)-curves. The divisor D is obtained by taking each curve with multiplicity 1, except the (-2)-curves drawn in boldface, which have multiplicity 2.

A configuration (II) occurs in Example 2 if each of the quartic curves A, B has a double point of type a_3 and if there is a line L tangent to A and B in these two double points (Figure 4). The divisor D is the lift of L to the fourfold cover

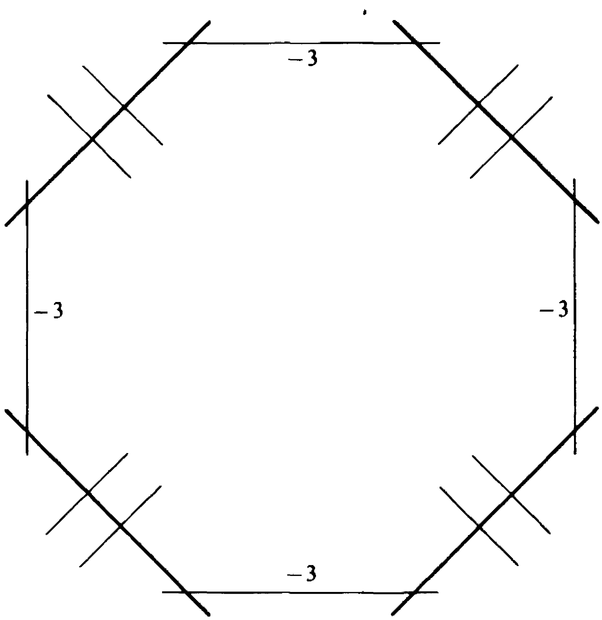

Figure 3. Configuration (II). $D^2 = KD = 4$.

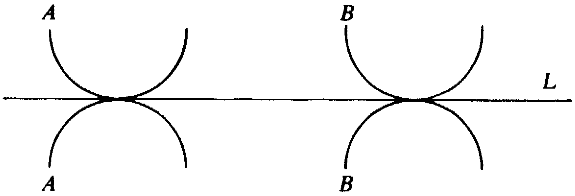

Figure 4

of the plane. The configuration (II) is a degeneration of the case when the two quartics have a common double tangent and we take the lift of the double tangent to the fourfold cover.

Many configurations can be constructed which are motivated by Example 3. Consider Figure 5. All curves are rational, except one which is elliptic. The

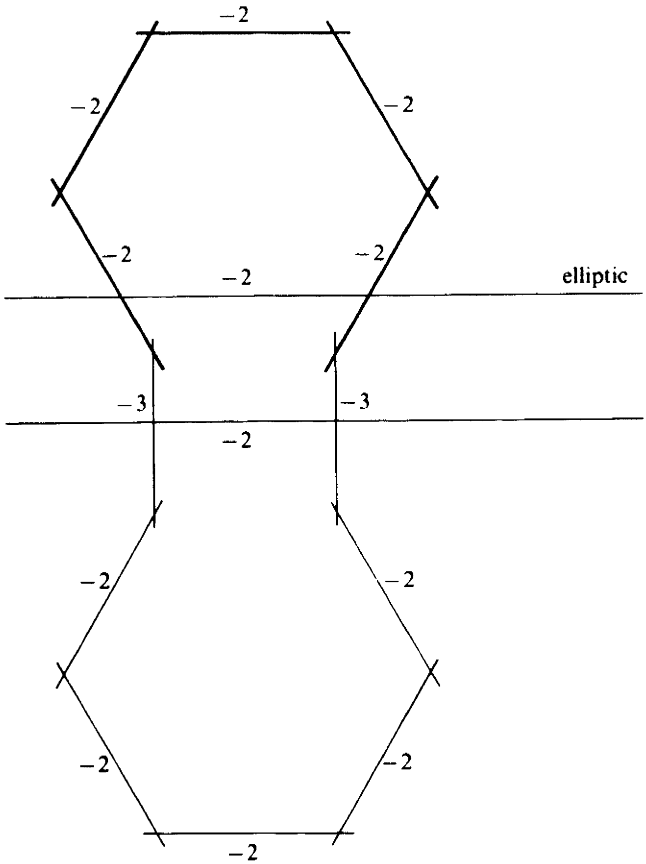

Figure 5. Configuration (III′). $D^2 = KD = 4$.

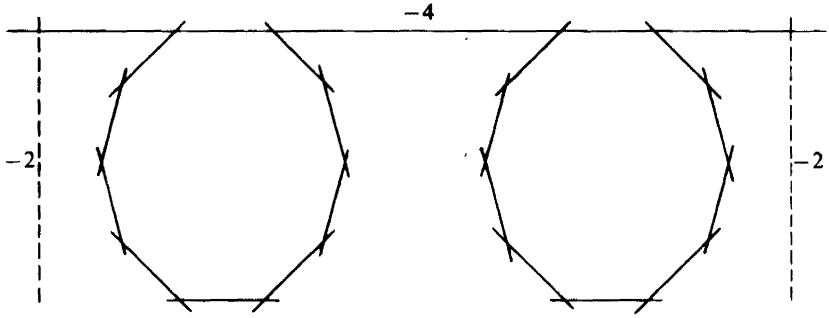

Figure 6. Configuration (III″). $D^2 = KD = 4$.

divisor D contains all curves with multiplicity 1, except the (-2)-curves drawn in boldface, which have multiplicity 2. Such a configuration arises if we take in Example 3 two lines L_1, L_2 on a nonsingular quadric belonging to different families of lines and assume that $L_1 \cap L_2$ is a double point of type a_5 of the branching curve C of bidegree $(6, 6)$, and that L_1 intersects C in 4 other points transversally, whereas L_2 passes through two other double points of C (of types a_1, a_5).

Let us now assume that in Example 3 the quadric is a cone and the ramification curve C does not pass through the vertex of the cone (Figure 6). Take a generating line L of the cone, and assume that it passes through two double points of type a_9 of C and intersects C transversally in two points. The lift of $2L$ leads to the above configuration (III″) of rational curves, one with self-intersection -4, all others with self-intersection -2. All curves have multiplicity 2 except the two curves indicated by a broken line, which have multiplicity 1 and are mapped to the vertex of the cone. In (III′) and (III″) we could use other double points, i.e. the lengths of the chains of (-2)-curves could be changed.

Take the configuration shown in Figure 7, consisting of 4 nonsingular rational curves (each with multiplicity 1). The existence of such a configuration on a surface proves minimality if $K^2 = 5$. It is motivated by Example 4 if a hyperplane intersects the quintic in a conic and 3 lines. All the transversal

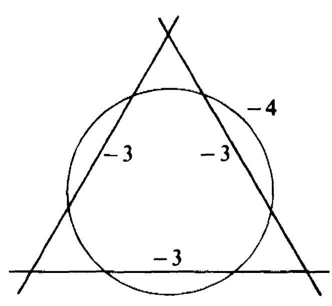

Figure 7. Configuration (IV). $D^2 = KD = 5$.

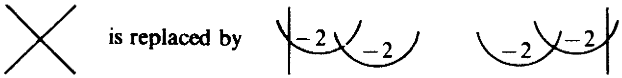

Figure 8

intersections may be replaced by chains of (-2)-curves, as shown in Figure 8. The modified configuration still satisfies $D^2 = KD = 5$.

There are cubic surfaces W for which the intersection of some plane with W consists of three lines L_1, L_2, L_3 which go through one point (Eckardt point). The three lines define an anticanonical divisor $L_1 + L_2 + L_3$ of W whose lift in the double cover (Example 5 of Section 1) is a configuration (V) (Figure 9) of three elliptic curves (with multiplicity 1). On a surface with $K^2 = 6$ the existence of a configuration of type (V) proves minimality.

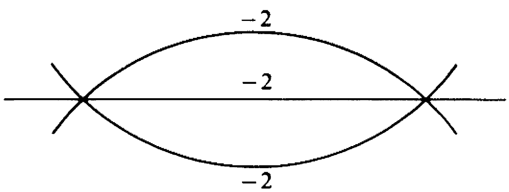

Figure 9. Configuration (V). $D^2 = KD = 6$.

Finally we consider Example 6 (Figure 10). The divisor D is of the form

$$D = 2C + \sum_{i=1}^{24} L_i,$$

where C is a nonsingular curve of genus 3 and there are twenty-four (-2)-curves intersecting C transversally. We have

$$D^2 = 2DK = 32.$$

The existence of a configuration (VI) on a surface with $K^2 = 8$ proves minimality. The divisor D and a double canonical divisor are homologous mod torsion. If the ramification curve in Example 6 is of the form $A + B$ where A and B are

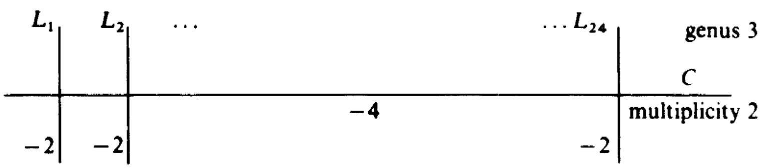

Figure 10. Configuration (VI).

nonsingular curves of degrees 4 and 6 intersecting transversally, then the lift of A gives a configuration (VI).

3. Hilbert Modular Surfaces

Let K be the real quadratic field of discriminant D over Q, and $\mathfrak{O}$ its ring of integers. The Hilbert modular group $G = SL_2(\mathfrak{O})/\{\pm 1\}$ operates on $H \times H$, where H is the upper half plane of C. The orbit space $H^2/SL_2(\mathfrak{O})$ can be compactified by finitely many cusps. The resulting surface $\overline{H^2/SL_2(\mathfrak{O})}$ has finitely many singularities, namely quotient singularities coming from elliptic fixed points and the cusps. Minimal desingularization gives the simply connected algebraic surface $Y(D)$. (For an introduction to Hilbert modular surfaces see [15] and [19].) The surface $Y(D)$ is rational for exactly 10 discriminants [19]; for all other discriminants it has a unique minimal model $Y_{\min}(D)$. On a nonrational $Y(D)$ certain curves can be blown down successively; they are described in [19]. The resulting surface $Y^0(D)$ is conjectured to be equal to $Y_{\min}(D)$. This has been proved in many cases (van der Geer and Van de Ven [10], Hirzebruch [16]). Freitag [7] and van der Geer [9] showed that for congruence subgroups Γ of G of sufficiently high level (Γ operates freely on H^2), the minimal resolution of the cusps of $\overline{H^2/\Gamma}$ leads to a minimal algebraic surface. The vector space of cusp forms of weight 2 for G is naturally isomorphic to $H^0(X, \Omega^2)$ if X is any nonsingular model for the algebraic surface $\overline{H^2/G}$. The same holds for a congruence subgroup Γ of G. (This result is due to Freitag; compare Hirzebruch [15, Section 3.5, Lemma]). Thus the canonical map of X in $P_{g-1}(C)$ (where $g = \dim H^0(X, \Omega^2)$) is induced by the "map" of H^2/G in $P_{g-1}(C)$ given by g linearly independent cusp forms of weight 2. Therefore the canonical map is especially interesting from the point of view of modular form theory as well. A cusp form of weight $2m$ with $m > 1$, in general, cannot be extended to the nonsingular model. The complete linear system of nonnegative m-fold canonical divisors is a birational invariant, but for $m > 1$, in general, it is smaller than the system of divisors of cusp forms of weight $2m$ (such a divisor can have components with negative multiplicities on the nonsingular model).

If the Hilbert modular surface $Y^0(D)$ is of general type, then $K^2 > 0$. (See Hirzebruch and Zagier [19].) Because it is simply connected (Švarčman [30]), a divisor D satisfying (1) in the Proposition of Section 2 is a canonical divisor. It is a nice program to write down such a divisor D in terms of explicitly known curves, to prove minimality in this way, and to get information on the canonical map by special properties of D. (Compare van der Geer and Zagier [8, 9, 11] for very similar studies.) As mentioned before, the surface $Y(D)$ is rational for exactly 10 discriminants; it is not rational and not of general type for 22 discriminants, the largest one being 165. In these 22 cases $Y^0(D)$ is minimal, namely a $K3$-surface or an honestly elliptic surface [19]. In all other cases, the surface is of general type. The calculations of [19] show that among those of general type there are exactly five discriminants with geometric genus 3 and seven discriminants with geometric genus 4. In these cases minimality can be proved and the nature of the canonical map determined:

Theorem. *The Hilbert modular surfaces $Y^0(D)$ of general type with $p_g = 3, 4$ and their values of K^2 are given by the following lists.*

$p_g = 3$:

D	89	97	124	141	168
K^2	2	2	2	2	4

$p_g = 4$:

D	101	104	109	113	133	156	161
K^2	4	4	4	6	4	4	4

In these cases $Y^0(D)$ is minimal. The canonical map is everywhere defined. For $p_g = 3$ and $D \neq 168$ it is a map of degree 2 on the projective plane ramified along a curve of degree 8 having as singularities only double points of type a_k (Moishezon surface, Example 1 of Section 1). For $D = 168$ the canonical map is of degree 4 (Example 2 in Section 1: the quartic A is reducible and consists of two conics; the quartic B is irreducible). For $p_g = 4$ and $D \neq 113$ the canonical map is of degree 2 onto a quadric surface in $P_3(C)$ which is nonsingular for $D \neq 104$ and is a cone for $D = 104$ (Example 3 in Section 1). For $D = 113$ the surface $Y^0(D)$ is mapped with degree 2 onto a cubic surface (Example 5 in Section 1) which has one singularity (of type A_3). On this cubic surface there are three lines passing through one point (Eckardt point). The cubic surface is uniquely determined by these two properties: One singularity (of type A_3), one Eckardt point.

We cannot give complete proofs here. Every case has to be studied individually. The following remarks will make it possible for the reader to check the results.

The Hilbert modular group G admits the Hurwitz–Maass extension G_m (see [19]). We take the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with entries in K such that $w = ad - bc$ is totally positive and $a/\sqrt{w}$, $b/\sqrt{w}$, $c/\sqrt{w}$, $d/\sqrt{w}$ are algebraic integers not necessarily in $\mathcal{O}$. The group G_m is the group of all these matrices divided by its center

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in K^* \right\}.$$

The group G_m/G is abelian of type $(2, \ldots, 2)$ with $t-1$ factors $Z/2$, where t is the number of primes dividing the discriminant D. The group G_m/G acts on $Y^0(D)$. It can be extended by the involution induced by the involution $\tau : (z_1, z_2) \to (z_2, z_1)$ of $H \times H$. This gives an abelian group M of type $(2, \ldots, 2)$ of order 2^t acting on $Y^0(D)$. Using results of Koll [26] and Hausmann [13] it is possible to determine the representation of M on $H^0(Y^0(D), \Omega^2)$, i.e. on the space of cusp forms of weight 2. Let M^0 be the subgroup of M consisting of the elements which operate on the cusps forms by $\pm$identity. Then the canonical map factors through $Y^0(D)/M^0$. For the discriminants in the theorem M^0 has order 2 except for $D = 168$, where it has order 4. The nontrivial element of M^0 ($D \neq 168$) is τ except for $D = 156$, where it is $\tau\alpha$. (Here α is the element of G_m/G represented by a matrix of determinant 13.) For $D = 168$ we have

$M^0 = \{1, \beta, \tau, \beta\tau\}$, where β is the element of G_m/G representable by a matrix of determinant 84.

The "explicitly known" curves which we use to construct canonical divisors are the curves coming from the resolution of the quotient singularities and the singularities at the cusps together with the modular curves F_N, also called skew-Hermitian curves. By a skew-Hermitian matrix we mean a matrix of the form

$$\begin{bmatrix} a_1\sqrt{D} & \lambda \\ -\lambda' & a_2\sqrt{D} \end{bmatrix} \qquad \text{where } \lambda \in \mathcal{O} \text{ and } a_1, a_2 \in Z.$$

The matrix is called primitive if there is no natural number >1 dividing a_1, a_2, λ. For given $N = a_1a_2D + \lambda\lambda'$ the curve F_n in H^2/G is defined to be the set of all points of H^2/G which have representatives $(z_1, z_2) \in H^2$ for which there exists a primitive skew-Hermitian matrix of determinant N such that

$$a_1\sqrt{D}\, z_1z_2 - \lambda' z_1 + \lambda z_2 + a_2\sqrt{D} = 0.$$

It can be shown that F_N defines a curve in $\overline{H^2/G}$ and also in $Y(D)$ and $Y^0(D)$. The curve F_N is nonempty if and only if none of the character values (D_q/N) equal -1 where D is the product of the t prime discriminants D_q.

The group M leaves F_N invariant and permutes the connectedness components of F_N. The surface $Y^0(D)$ is obtained by blowing down all components of F_1, F_2, F_3, F_4, and F_9 (if $3 \mid D$), provided these curves are not empty, together with the curves into which the quotient singularities lying on F_1 and F_2 were resolved. The intersection hehavior of the curves F_N is completely known (see Hirzebruch and Zagier [20] for D a prime, Hausmann [13] in general). It is also known how the F_N pass through the curves of the resolution of the cusps. The number of connectedness components of F_N was determined in [13]. Also the genus $g(C)$ and the value KC can be calculated for each connectedness component C of F_N (Hirzebruch and Zagier [15, 19]), at least if N satisfies certain number-theoretical conditions which are always fulfilled in the cases we need. Therefore all the information needed to construct canonical divisors is available.

If an element α of G_m/G can be represented by a primitive matrix

$$\begin{bmatrix} \lambda' & -a_2\sqrt{D} \\ a_1\sqrt{D} & \lambda \end{bmatrix}$$

with determinant N dividing D, then the curve F_N is pointwise fixed under $\alpha\tau$. Using this remark, one can determine the ramification curve C in the above cases. For $D \equiv 1 \bmod 4$ the curve C equals F_D. For $D \equiv 0 \bmod 4$ ($D \neq 156$, $D \neq 168$) the curve C is the union of F_D and $F_{D/4}$. For $D = 156$ the ramification curve is $F_{13} \cup F_{52}$. For $D = 168$ the curve $F_{168} \cup F_{42}$ is pointwise fixed under τ, and F_{84} pointwise fixed under $\beta\tau$.

We now give some examples. All canonical divisors constructed will be invariant under M with one exception, and all of them arise in the way indicated in Section 2. They not only help to prove minimality, but are also important to determine the nature of the canonical map.

$D = 124$

We construct a configuration of type (I') involving F_{10} (as curve $\tilde{L}$), F_8, F_9, F_{16}, and curves coming from the singularities of H^2/G. We have $k_1 = k_2 = 5$ and $k_3 = 7$.

$D = 141$

We construct a configuration of type (I″) involving F_{21} as curve $\tilde{L}$, using F_7 and curves coming from the singularity at the unique cusp. We have $k_1 = k_2 = 3$.

$D = 168$

We get a configuration of type (II) involving F_7 and F_{18}. The two conics are the images of F_{42} and F_{168}; the quartic B is the image of of F_{84}. The two conics touch each other (double point a_3 of A); the quartic B touches each conic (double point a_3 of $A \cup B$) and has itself a double point a_3.

$D = 133$

We obtain a configuration of type (III') involving F_7, F_9, F_{11} where one component of F_{11} is the elliptic curve. This divisor is not invariant under the nontrivial element of G_m/G.

$D = 104$

The configuration is of type (III″). The curve with self-intersection -4 is F_{10}. The chains of (-2)-curves come from the two cusp singularities. The two (-2)-curves which map to the vertex of the cone are the resolutions of the two quotient singularities of order 2 lying on F_{10}.

$D = 113$

The curves F_{15}, F_{16}, F_{18} are the three elliptic curves of configuration (V). The two intersection points are special points (Hirzebruch and Zagier [20]) with the quadratic form $15u^2 + 16uv + 13v^2$ of discriminant -7×113, which represents 15, 16, and 18. The A_3-singularity of the cubic comes from 2 chains of (-2)-curves of length 3 interchanged under the involution τ, which occur in the resolution of the unique cusp of $\overline{H^2/G}$. As Wall pointed out to me, there is a one-parameter family of cubics with exactly one singular point (of type A_3), and in this family there is a unique cubic with an Eckardt point. In fact this cubic can be written down with respect to suitable homogeneous coordinates as

$$xyw + x^3 - xz^2 - yz^2 + y^3 = 0.$$

The cubic has 10 lines. They are the images of $F_{11}, F_{13}, F_{14}, F_{15}, F_{16}, F_{18}, F_{25}$, of two pairs of (-3)-curves of the cusp resolution, and of the pair of (-3)-curves which arise from the two quotient singularities of order 3 lying on F_7.

The singular point of the cubic surface is $(x, y, z, w) = (0, 0, 0, 1)$. The plane $w = 3(x + y)$ intersects the cubic surface in the three lines given by $x + y = 0$, $z = x + y$, $z = -(x + y)$, which pass through the Eckardt point $(1, -1, 0, 0)$.

The ramification curve C (corresponding to F_{113}) has on the cubic surface 5 singularities of type a_1, 3 of type a_2, 2 of type a_3, 1 of type a_8. The genus of C equals 3.

4. More Examples

It is interesting to study the canonical map for all Hilbert modular surfaces $Y^0(D)$ which fall into the range of Horikawa's classification $(2p_g - 4 < K^2 < 2p_g - 2)$. These are only finitely many. For large D we have $K^2 \sim 8p_g$. Since K^2 is even for $Y^0(D)$, we should investigate the discriminants with $K^2 = 2p_g - 4$ or $K^2 = 2p_g - 2$. It turns out that $K^2 = 2p_g - 4$ happens only if $p_g = 3$ or 4. These discriminants were treated in Section 3. We have $K^2 = 2p_g - 2$ for $D = 168$ $(p_g = 3)$ and $D = 113$ $(p_g = 4)$. Also these cases were studied in Section 3. Otherwise $K^2 = 2p_g - 2$ if and only if $D = 129$, 136, 184 $(p_g = 5)$, $D = 145$, 149 $(p_g = 6)$, $D = 204$ $(p_g = 7)$.

We only consider the prime discriminant $D = 149$. It turns out that $Y^0(149)$ is minimal. It is the double cover of a del Pezzo surface W of degree 5 in $P_5(C)$ ramified along a 4-fold anticanonical curve C of W corresponding to F_{149}. The curve C on W has 9 singularities of type a_1, 8 singularities of type a_2, 3 of type a_3, and 1 of type a_{10}. It has genus 3.

The surface W contains 10 lines (exceptional curves). They are the images of $F_{19}, F_{20}, F_{22}, F_{24}, F_{28}, F_{36}$, of two pairs of (-3)-curves coming from the resolution of the cusp, and of the two pairs of (-3)-curves coming from resolving the four quotient singularities of order 3 lying on F_6 and F_{16}.

The surface $Y^0(149)$ has exactly 44 nonsingular rational curves of self-intersection -2 which, together with the above-mentioned 14 irreducible curves (mapped to the 10 lines of the del Pezzo surface), generate a vector space of algebraic cycles of dimension 53. This vector space coincides with the space of algebraic cycles generated by the curves F_N on $Y^0(149)$ and the curves coming from resolving the singularities of $\overline{H^2/G}$. In fact, the Picard number of $Y^0(149)$ is greater than or equal to 54. We have $h^{1,1} = 60$.

We now consider two examples of congruence subgroups.

Let G be the Hilbert modular group for $D = 13$ and Γ the principal congruence subgroup for the ideal (2) of $K = Q(\sqrt{13})$. Let Y be the surface obtained by resolving the 5 cusps of $\overline{H^2/\Gamma}$. For Y we have $p_g = 4$ and $K^2 = -5$. However, on Y the curve F_1 has 10 components, all of which are exceptional curves. Blowing down these ten curves, we obtain a new surface with $K^2 = 5$. For each of the five cusps van der Geer and Zagier construct in [11] a configuration (IV) (with the three transversal intersections of the three lines in (IV) replaced by a chain of two (-2)-curves in each case). The 3 lines and the (-2)-curves correspond to the resolution of the cusp; the conic is one of the five components of F_4. In this way the authors of [11] obtain 5 cusp forms $s_0, s_1, \ldots, s_4$ of weight 2 satisfying $s_0 + s_1 + \cdots + s_4 = 0$ and prove that the canonical map ι_K is a holomorphic mapping of degree 1 of Y onto the quintic surface in $P_4(C)$ defined by the equations $\sigma_1 = 0$, $2\sigma_5 - \sigma_2\sigma_3 = 0$, where the 15 singularities of the quintic are of type A_2. They are images of the 15 chains of (-2)-curves of length 2 mentioned before. Otherwise ι_K is biholomorphic. (σ_i is the ith elementary symmetric function in the s_j.)

Let G be the Hilbert modular group for $D = 8$. The prime 7 splits in $K = Q(\sqrt{2})$. We have $(7) = \mathfrak{g}\mathfrak{g}'$. Consider the principal congruence subgroup Γ of G for the ideal $\mathfrak{g} = (3 + \sqrt{7})$. The group G/Γ is isomorphic to $PSL_2(F_7)$, the famous simple group G_{168} of order 168. The group Γ acts freely on H^2. The surface H^2/Γ has to be compactified by eight cusps corresponding to the points of $P_1(F_7)$. The resolution of each of the eight cusps of $\overline{H^2/\Gamma}$ consists of a cycle of six rational nonsingular curves with self-intersection number -4, -2, -4, -2, -4, -2. We denote the algebraic surface obtained by resolving the 5 cusps by Y. The normalized Euler volume of H^2/G equals $2\zeta_K(-1) = \frac{1}{6}$. Therefore the Euler number of H^2/Γ is $168/6 = 28$. For the Euler number of Y we get

$$e(Y) = 28 + 8 \times 6 = 76.$$

Since $Q(\sqrt{2})$ has a unit of negative norm, the arithmetic genus of Y equals $\frac{1}{4} e(H^2/\Gamma) = 7$. (See Hirzebruch [15, Section 3.6].)

The curve F_1 of Y is irreducible, nonsingular, and isomorphic to $\overline{H/\Gamma(7)}$ where $\Gamma(7)$ is the principal congruence subgroup of $SL_2(Z)/\{\pm 1\}$ of level 7. It has 24 cusps. The Euler number is given by the formula

$$e(F_1) = -\tfrac{1}{6} \cdot 168 + 24 = -4.$$

Here $-\frac{1}{6}$ is the normalized Euler volume of $H/SL_2(Z)$. Hence F_1 has genus 3. We also consider the curve F_2 of Y. It is irreducible and nonsingular. The quotient of F_2 by $G_{168} = G/\Gamma$ is the curve F_2 in $\overline{H^2/G}$ whose nonsingular model is $\overline{H/\Gamma_0^*(2)}$; see [15, Section 4.1]. Therefore

$$e(F_2) = -\tfrac{1}{6} \cdot \tfrac{3}{2} \cdot 168 + 24 = -18.$$

Thus F_2 has genus 10. It can be shown that F_1 and F_2 pass through each of the eight resolved cusps of Y as shown in Figure 11. The intersection of a canonical divisor K of Y with F_1 or F_2 can be calculated [15, Section 4.3, (19)]. This intersection number equals the normalized Euler volume of the curve multiplied by -2 decreased by the sum of the intersection numbers with the curves in the resolutions of the cusps.

We have

$$KF_1 = +\tfrac{1}{3} \cdot 168 - 48 = 8,$$

$$KF_2 = +\tfrac{1}{2} \cdot 168 - 72 = 12,$$

$$F_1F_1 = -4 \quad \text{and} \quad F_2F_2 = 6.$$

The curve F_1 (multiplicity 2) and the twenty-four (-2)-curves of the eight cusps of Y constitute a configuration (VI) of Section 2.

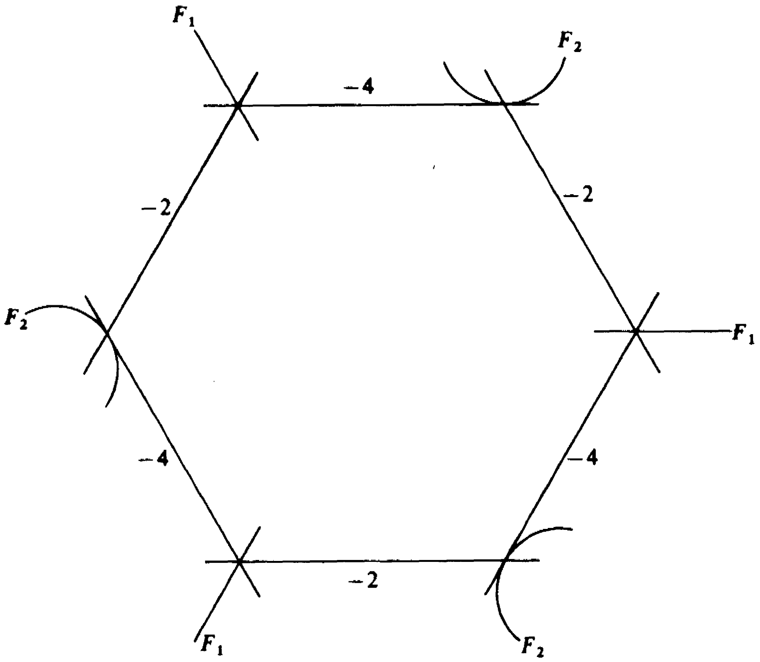

Figure 11

We have proved that Y is a minimal algebraic surface with $\chi = 7$ and Euler number 76. By Noether's formula $K^2 = 8$. We know that Y is simply connected. Hence $p_g = 6$. According to Horikawa's result [22] on minimal algebraic surfaces with $K^2 = 2p_g - 4$, the canonical map is of degree 2 onto a surface of degree 4 in $P_5(C)$. This is either the Veronese surface (projective plane) or a ruled surface. Since G_{168} acts on Y, it also acts on the image surface in $P_5(C)$. But a ruled surface does not admit such an action. Therefore Y is a double cover of the projective plane branched along a curve of degree 10 (Example 6 in Section 1).

We have seen that

$$D_1 = 2F_1 + \sum_{i=1}^{24} L_i$$

is a twofold canonical divisor of Y (where the L_i are the twenty-four (-2)-curves occuring in the cusps). We can show in the same way that

$$D_2 = 2F_2 + \sum_{i=1}^{24} L_i$$

is a threefold canonical divisor. Under the canonical map $\iota_K : Y \to P_5(C)$, where $\iota_K(Y) \simeq P_2(C)$, the image of $D_1 \cup D_2$ must be contained in the curve of ramification, because D_1, D_2 are not divisible by 2, but have F_1 and F_2 as components of multiplicity 2.

The group G_{168} acts on Y and also on $\iota_K(Y) \simeq P_2(C)$. This is the well-known action of G_{168} on $P_2(C)$, because this action, which is described by Weber [31, 15. Abschnitt] is essentially unique.

The action of G_{168} on the projective plane has a unique invariant curve A of degree 4 and a unique invariant curve B of degree 6. These must be the images of F_1 and F_2. The curve A is the famous curve of genus 3 studied by Felix Klein, which has G_{168} as automorphism group. The curves A and B are nonsingular. They intersect transversally in 24 points which are the flexes of A. The 24 tangents of inflexion of A are arranged in 8 "triangles": The tangent T_1 of A in a flex p_1 of A intersects A in a flex p_2 transversally; the tangent T_2 of A in p_2 intersects A in a flex p_3; the tangent T_3 of A in p_3 intersects A in p_1. The line T_1 is tangent to B in p_2 with intersection multiplicity 5; the same holds for T_2 in p_3 and for T_3 in p_1.

Under the canonical map the (-4)-curves of a resolution cycle of a cusp go to such a triangle T_1, T_2, T_3. The (-2)-curves are mapped to the flexes p_1, p_2, p_3.

The canonical involution σ on Y carries a (-4)-curve to another (-4)-curve which does not belong to any cuspidal resolution. This shows that σ is not modular, i.e., it is not induced by an automorphism of H^2. Because G_{168} is a maximal finite automorphism group of $P_2(C)$, the automorphism group of Y is the direct product $G_{168} \times Z/2$, where σ is the nontrivial element of $Z/2$. We collect the above information in the following theorem.

Theorem. *The (desingularized) Hilbert modular surface Y for the field $Q(\sqrt{2})$ and the principal congruence subgroup of the Hilbert modular group with respect to an ideal of norm 7 is a minimal algebraic surface with $p_g = 6$ and $K^2 = 8$. Under the canonical map it can be realized as the desingularized double cover of the Veronese surface (identified with $P_2(C)$) ramified along the curve $A \cup B$ of degree 10, where A and B are the unique invariant curves of degree 4 and 6 respectively for the action of G_{168} on $P_2(C)$. The full automorphism group of Y is $G_{168} \times Z/2$, where the canonical involution of Y is the nontrivial element of $Z/2$. This involution is not modular.*

5. Remarks on the Symmetric Hilbert Modular Group and on "Modular" Modular Forms

As mentioned in Section 3, the surface $Y^0(D)$ admits an action of the group M of order 2^t. If D is a prime p ($p \equiv 1 \bmod 4$), then M consists only of the involution τ induced by $(z_1, z_2) \to (z_2, z_1)$. It would be interesting to study the canonical map for the surfaces $Y^0(D)/M$. If, for example, the minimal model of such a surface is a Moishezon surface, then the involution on it is nonmodular. However, only some results on $Y^0(p)/\tau$ (p prime) are known (Hirzebruch and Van de Ven [18]). The surface $Y^0(p)/\tau$ is rational for exactly 24 primes. If it is not rational, then the surface $Y_\tau^0(p)$ is defined by blowing down on $Y^0(p)/\tau$

certain curves F_N and curves coming from the resolution of cusp and quotient singularities (Hirzebruch [16]).

It is conjectured that $Y_\tau^0(p)$ is minimal.

The geometric genus of $Y_\tau^0(p)$ equals 3 if and only if $p = 313, 653, 677, 773$. In these cases $K^2 = 2$, the surface is minimal and is a Moishezon surface [18].

The geometric genus of $Y_\tau^0(p)$ equals 4 if and only if $p = 337, 401, 541, 797$. In these cases $K^2 = 5, 6, 6, 5$. Are these surfaces minimal? If yes, is the canonical map holomorphic of degree 1 onto a quintic surface for $p = 337, 797$ and a double cover of a cubic surface for $p = 401, 541$?

The construction of a canonical divisor on $Y^0(D)$ by curves F_N and curves coming from the singularities gives a cusp form f of weight 2 for the Hilbert modular group G of $Q(\sqrt{D})$ whose zero divisor (f) on H^2/G consists only of modular curves F_N. Such a cusp form could be called "modular". The same remark applies to $Y_\tau^0(p)$, where such canonical divisors give "modular" cusps forms of weight 2 satisfying $f(z_1, z_2) = -f(z_2, z_1)$. (See [18, p. 147, Remark 2].) If one had a general theorem which guarantees the existence of a "modular" cusp form f of weight 2 (which is supposed to be skew-symmetric if one considers $Y_\tau^0(p)$), then the problem of minimality would be solved, because then all exceptional curves of a desingularization Y of $\overline{H^2/G}$ or $\overline{(H^2/G)/\tau}$ must be contained in the canonical divisor defined by f on Y, and this canonical divisor consists only of curves F_N and curves coming from singularities.

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