<u>Euler, Riemann, Riemann-Roch</u>

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Remark: Transparency no. 14 was shown after transparency no. 6

REMARQUES SUR UN BEAU RAPPORT ENTRE LES SERIES DES PUISSANCES TANT DIRECTES QUE RECIPROQUES[1]

Commentatio 352 indicis Enestroemiani
Mémoires de l'académie des sciences de Berlin [17] (1761), 1768, p. 83—106
Lu en 1749

1. Le rapport que je me propose de développer ici regarde les sommes de ces deux séries infinies générales

$$\odot \quad 1^m - 2^m + 3^m - 4^m + 5^m - 6^m + 7^m - 8^m + \text{etc.},$$

$$☽ \quad \frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \frac{1}{7^n} - \frac{1}{8^n} + \text{etc.},$$

$$\varphi(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots$$

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

$$\varphi(s) = (1 - 2^{1-s})\zeta(s)$$

$$1 - x + x^2 - x^3 + \text{etc.} = \frac{1}{1+x},$$

$$1 - 2x + 3x^2 - 4x^3 + \text{etc.} = \frac{1}{(1+x)^2},$$

$$1 - 2^2x + 3^2x^2 - 4^2x^3 + \text{etc.} = \frac{1-x}{(1+x)^3},$$

$$1 - 2^3x + 3^3x^2 - 4^3x^3 + \text{etc.} = \frac{1-4x+xx}{(1+x)^4},$$

$$1 - 2^4x + 3^4x^2 - 4^4x^3 + \text{etc.} = \frac{1-11x+11xx-x^3}{(1+x)^5},$$

$$1 - 2^5x + 3^5x^2 - 4^5x^3 + \text{etc.} = \frac{1-26x+66xx-26x^3+x^4}{(1+x)^6},$$

$$1 - 2^6x + 3^6x^2 - 4^6x^3 + \text{etc.} = \frac{1-57x+302xx-302x^3+57x^4-x^5}{(1+x)^7}$$

etc.,

d'où l'on tire pour les séries de notre premiere espece, en prenant $x = 1$, les sommes suivantes:

$$1-2^0+3^0-4^0+5^0-6^0+\text{etc.}=\frac{1}{2},$$

$$1-2+3-4+5-6+\text{etc.}=\frac{1}{4},$$

$$1-2^2+3^2-4^2+5^2-6^2+\text{etc.}=0,$$

$$1-2^3+3^3-4^3+5^3-6^3+\text{etc.}=-\frac{2}{16},$$

$$1-2^4+3^4-4^4+5^4-6^4+\text{etc.}=0,$$

$$1-2^5+3^5-4^5+5^5-6^5+\text{etc.}=+\frac{16}{64},$$

$$1-2^6+3^6-4^6+5^6-6^6+\text{etc.}=0,$$

$$1-2^7+3^7-4^7+5^7-6^7+\text{etc.}=-\frac{272}{256},$$

$$1-2^8+3^8-4^8+5^8-6^8+\text{etc.}=0,$$

$$1-2^9+3^9-4^9+5^9-6^9+\text{etc.}=+\frac{7936}{1024}$$

etc.

$$\frac{P_n(z)}{(1-z)^{n+1}} = \sum_{k=1}^{\infty} k^n z^{k-1}$$

$P_0(z) = 1$, $P_n(z)$ has degree $n-1$ for $n \geq 1$

$P_1(z) = 1$, $P_2(z) = 1 + z$

$P_3(z) = 1 + 4z + z^2$

$P_4(z) = 1 + 11z + 11z^2 + z^3$

$$P_n(z) = \sum_{k=0}^{n-1} W_{n,k} z^k$$

$$f(x) = \sum_{n=1}^{\infty} P_n(z) \frac{x^n}{n!} = \frac{e^{(1-z)x} - 1}{1 - ze^{(1-z)x}}$$

$$z = 1 \quad \Longrightarrow \quad f(x) = \frac{x}{1-x} \quad , \quad P_n(1) = n!$$

$$z = -1 \quad \Longrightarrow \quad f(x) = \mathrm{tgh}(x)$$

$$P_n(-1) = \mathrm{tgh}^{(n)}(0)$$

$$P_n(-1) = 0 \quad \text{for} \quad n \quad \text{even} \quad \geq 2$$

$$P_n(-1) = (-1)^{k-1}\,\mathrm{tg}^{(2k-1)}(0) \quad \text{for} \quad n = 2k-1$$

<u>Euler</u>

$$\varphi(-n) = 0 \quad \text{for} \quad n \quad \text{even} \quad \geq 2$$

$$\varphi(1-2k) = (-1)^{k-1}\,\mathrm{tg}^{(2k-1)}(0)/2^{2k}$$

$$\varphi(0) = \frac{1}{2}$$

$$\mathrm{tg}^{(2k-1)}(0) = t_{2k-1} > 0$$

$$1,\ 2,\ 16,\ 272,\ 7936,\ \ldots$$

Institutiones calculi differentialis II, 1755

$$\frac{1}{1-u^2}+\frac{1}{4-u^2}+\frac{1}{9-u^2}+\frac{1}{16-u^2}+\text{etc.}=\frac{1}{2uu}-\frac{\pi}{2u}\cot \pi u.$$

Resolvantur singulae istae fractiones in series

$$\frac{1}{1-u^2}=1+u^2+u^4+u^6+u^8+\text{etc.}$$

$$\frac{1}{4-u^2}=\frac{1}{2^2}+\frac{u^2}{2^4}+\frac{u^4}{2^6}+\frac{u^6}{2^8}+\frac{u^8}{2^{10}}+\text{etc.}$$

$$\frac{1}{9-u^2}=\frac{1}{3^2}+\frac{u^2}{3^4}+\frac{u^4}{3^6}+\frac{u^6}{3^8}+\frac{u^8}{3^{10}}+\text{etc.}$$

$$\frac{1}{16-u^2}=\frac{1}{4^2}+\frac{u^2}{4^4}+\frac{u^4}{4^6}+\frac{u^6}{4^8}+\frac{u^8}{4^{10}}+\text{etc.}$$

etc.

$$2\sum_{k=1}^{\infty} \zeta(2k)u^{2k} = 1-\pi u \cot \pi u$$

$$\sum_{k=1}^{\infty} \frac{B_k}{(2k)!} u^{2k} = 1-\frac{u}{2}\cot\frac{u}{2}$$

$$B_1=\frac{1}{6}, \ldots, B_{15}=\frac{8615841276005}{14322}$$

$$\operatorname{tg}(x)=\sum_{k=1}^{\infty}\frac{t_{2k-1}}{(2k-1)!}\;x^{2k-1}\;=\;\cot x\;-2\cot 2x$$

$$\varphi(2k)=\frac{\pi^{2k}}{(2k-1)!\;2^{2k}}\cdot\frac{(2^{2k-1}-1)}{(2^{2k}-1)}t_{2k-1}$$

$$\frac{\varphi(1-2k)}{\varphi(2k)}=\frac{-(2k-1)!\;(2^{2k}-1)}{(2^{2k-1}-1)\pi^{2k}}(-1)^{k}$$

Par cette raison je hazarderai la *conjecture suivante*, que, quel que soit l'exposant *n*, cette équation ait toujours lieu

$$\frac{1-2^{n-1}+3^{n-1}-4^{n-1}+5^{n-1}-6^{n-1}+\text{etc.}}{1-2^{-n}+3^{-n}-4^{-n}+5^{-n}-6^{-n}+\text{etc.}}=$$

$$=\frac{-1\cdot2\cdot3\cdots(n-1)(2^{n}-1)}{(2^{n-1}-1)\,\pi^{n}}\cos.\frac{n\pi}{2}.$$

$$\frac{\varphi(1-n)}{\varphi(n)}=\frac{-\Gamma(n)(2^{n}-1)}{(2^{n-1}-1)\pi^{n}}\;\;\cos\frac{n\pi}{2}$$

Some combinatorics

$$P_{n+1}(z) = P_n(z)(1+nz) + z(1-z)P_n'(z)$$

$$W_{n+1,k} = (k+1)\, W_{n,k} + (n-k+1)\, W_{n,k-1}$$

This proves by induction

$W_{n,k}$ is the number of permutations of $\{1,2,\ldots n\}$ with k ascents.

The permutation σ has k ascents if $\sigma(i+1) > \sigma(i)$ holds for exactly k numbers $i \in \{1,2,\ldots n\}$.

X compact complex manifold, $\dim_{\mathbb{C}} X = d$

total Chern class

$$c = 1 + c_1 + \ldots + c_d \quad \text{with} \quad c_i \in H^{2i}(X, \mathbb{Z})$$

Chern numbers

$$c_{r_1} c_{r_2} \ldots c_{r_s}[X] \quad \text{for} \quad r_1 + r_2 + \ldots + r_s = d$$

genus u :

$u(X) \in \mathbb{Q},\ \mathbb{Q}[y],\ \ldots$

properties (definition)

1) $u(X + Y) = u(X) + u(Y)$

2) $u(X \times Y) = u(X) \cdot u(Y)$

3) $u(X)$ depends only on the Chern numbers of X

A genus u is given by a power series

$$Q(x) = \frac{x}{f(x)} = 1 + \ldots$$

where

$$f(x) = \sum_{n\geq 1}^{\infty} a_n \frac{x^n}{n!} \quad , \quad a_1 = 1$$

Write formally

$$1 + c_1 + \ldots + c_d = (1+\gamma_1)(1+\gamma_2)\ldots(1+\gamma_d)$$

where $\gamma_i \in H^2(?, \mathbb{Z})$

We have ($\dim_{\mathbb{C}} X = d$) :

$$u(X) = Q(\gamma_1)Q(\gamma_2)\ldots Q(\gamma_d)[X]$$

A abelian variety, principally polarized:

$A = \mathbb{C}^n / \Gamma$

A has a Kähler metric which defines $x \in H^2(A, \mathbb{Z})$

of type $(1,1)$ and $x^n[A] = n!$

Θ theta divisor:

choose line bundle L with $c_1(L) = x$.

Then $H^0(A, \mathcal{O}(L))$ is 1-dimensional and determines Θ.

Generically Θ is smooth.

Θ is an $(n-1)$-dimensional projective algebraic manifold whose tangent bundle plus the line bundle L is trivial.

If a genus u is given by

$$f(x) = \sum_{n\geq 1}^{\infty} a_n \frac{x^n}{n!} \quad ,$$

then

$$a_n = u(\Theta)$$

<u>Remark:</u> All Chern numbers of Θ are equal to each other. All have the value $(-1)^{n-1} n!$

The χ_y-genus

$\dim_{\mathbb{C}} X = d$.

$$\chi^p(X) = \chi(X, \mathcal{O}(\Lambda^p T^*))$$

If X is Kähler,

$$\chi^p(X) = \sum_{q=0}^{d} (-1)^q h^{p,q}$$

Define

$$\chi_y(X) = \sum_{p=0}^{d} \chi^p(X) y^p$$

One of the main results of RRH (1953) is that $\chi_y(X)$ is the genus with power series

$$* \qquad f(x) = -\frac{e^{-(1+y)x} - 1}{1 + ye^{-(1+y)x}}$$

For compact complex manifolds one needs the index theory of Atiyah-Singer.

The $f(x)$ in * is essentially the same as the $f(x)$ of Euler polynomials

$$f(x)=\sum_{n=1}^{\infty} P_n(z)\frac{x^n}{n!}=\frac{e^{(1-z)x}-1}{1-ze^{(1-z)x}}$$

<u>Euler's</u> $f(x)$ <u>gives the genus</u>

$$(-1)^d \chi_{-z}(X)$$

In particular for $X=\Theta$ of dimension $n-1$

$$P_n(z)=(-1)^{n-1}\chi_{-z}(\Theta)$$

$$W_{n,k}=(-1)^{n-1-k}\chi^k(\Theta)$$

For $z = 1$ we have

$$P_n(1) = n! = \sum_{k=o}^{n-1} W_{n,k}$$

$(-1)^{n-1} n!$ = Euler number of Θ

$$P_n(0) = 1$$

$$(-1)^{n-1} = \chi^0(\Theta) =$$

arithmetic genus = Todd genus

For $n = 2k - 1$

$$\sum_{k=0}^{n-1} W_{n,k} (-1)^k = \text{tgh}^{(2k-1)}(0)$$

= signature of Θ of
dimension $2k - 2$.

A abelian variety, Θ smooth theta divisor

$\Theta \subset A$, $\dim \Theta = n-1$, $\dim A = n$

$$e(\Theta) = (-1)^{n-1} n!$$

Lefschetz theorem for hyperplane sections implies for rational cohomology:

$$H^i(A) \to H^i(\Theta) \text{ bijective for } i < n-1$$

$$H^i(A) \to H^i(\Theta) \text{ injective for } i = n-1$$

For the Betti numbers:

$$b_i(A) = \binom{2n}{i} = b_i(\Theta) \text{ for } i < n-1$$

By Poincaré duality

$$b_{2n-2-i}(\Theta) = b_i(\Theta)$$

All Betti numbers of Θ are known

except $b_{n-1}(\Theta)$

Using $e(\Theta) = (-1)^{n-1} n!$ gives

$b_{n-1}(\Theta) - b_{n-1}(A) = n! - C_n$

where $C_n = b_n(A) - b_{n+1}(A) = \binom{2n}{n} / n+1$

is the n-th Catalan number.

By Lefschetz all Hodge numbers of Θ

are known except $h^{p,n-1-p}(\Theta)$

Since

$$(-1)^{n-1-p} \sum_{q=0}^{n-1} (-1)^q h^{p,q}(\Theta) = W_{n,p}$$

also $h^{p,n-1-p}$ is known.

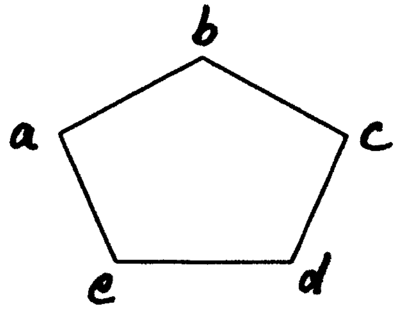
Euler to Goldberg Aug./Sept. 1751.

Nun ist die Frage generaliter: da ein polygonum von n Seiten durch $n-3$ diagonales in $n-2$ triangula zerschnitten wird, auf wie vielerlei verschiedene Arten solches geschehen könne. Setze ich nun die Anzahl dieser verschiedenen Arten $=x$, so habe ich per Induktionem gefunden

wann n	$= 3, 4, 5, 6, 7, 8, 9, 10$
so sit x	$= 1, 2, 5, 14, 42, 132, 429, 1430.$

Euler's x is C_{n-2}

Ein Fünfeck



wird durch 2 Diagonales in 3 Triangula geteilet,

und solches kann auf 5erlei verschiedene Arten geschehen,

nämlich durch die Diagonales

I. *ac*,	II. *bd*,	III. *ca*,	IV. *db*,	V. *ec*,
ad.	*be*.	*ce*.	*da*.	*eb*.