Differentiable Manifolds and Quadratic Forms

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Errata  (§0 - §3)

Page    line  should read
0-3    4     $g = (g,W)$ [instead of $q = (q,W)$]
0-6    3b    $a_{ll} \notin A^*$
0-17    1     $a_{1}a_{1}^{2} + \ldots$
A-5    9-12  $p \equiv 1 \pmod{4}$       $p \equiv 3 \pmod{4}$
          1
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1-3    5b    $-\det + 1$
1-4    6-7    $\ldots = axy$, where $a = \det f (\ldots\ldots)$ is odd.
1-5    12    $-\det + 1$
3-7    3     §4
3-7    12    the first "$z_2$" should read "$\bar{z}_2$".
0. Quadratic forms.

Let \( A \) be an integral domain. By a lattice over \( A \) (or simply an \( A \)-lattice) we shall mean a finitely generated, free, unitary \( A \)-module. The terms base, rank, etc., will be employed in the usual fashion. Occasionally, we shall consider \( A \) itself as an \( A \)-lattice of rank 1 (the module structure being defined by the ring operations in \( A \)). For a given \( A \)-lattice, \( V \), the module \( \text{Hom}_A(V,A) = V' \) is again a lattice of the same rank. We shall call \( V' \) the dual lattice of \( V \). There is a pairing (called the Kronedker product) from \( V' \times V \) into \( A \) defined by \( \langle x', x \rangle = x'(x) \) for \( x' \in V' \) and \( x \in V \).

For our purpose, a quadratic form, \( f = (f,V) \), over \( A \) may be defined to be a bilinear, symmetric pairing

\[
f: V \times V \to A,
\]

where \( V \) is an \( A \)-lattice and, in this case, we shall also say that \( f \) is defined on \( V \). To such a form, \( f \) there corresponds a linear map \( \mathcal{f}: V \to V' \), to be called the correlation associated with \( f \), given by

\[
(0.1) \quad \langle \mathcal{f}(x), y \rangle = f(x,y) ,
\]

for all \( x \) and \( y \) in \( V \). The form, \( f \), will be termed non-degenerate if its associated correlation, \( \mathcal{f} \), is
injective (i.e. a monomorphism). This property is also characterized by

\[(0.2) \quad f(x, y) = 0 \quad \text{for all} \quad y \in V \Rightarrow x = 0.\]

If a base \( (e_i)_{1 \leq i \leq r} \) is chosen in \( V \), there is a dual base \( (e'_i)_{1 \leq i \leq r} \) in \( V' \) characterized by the property that \( \langle e'_i, e_j \rangle = \delta_{ij} \) (the Kronecker deltas).

Put \( f(e_i) = \sum_j \alpha_{ij} e'_j \), then \( \alpha_{ij} = f(e_i, e_j) \). The matrix \( M = M_f = (\alpha_{ij}) \) will be called the matrix of \( f \) (with respect to \( (e_i) \)), and its rank, which is independent of the base chosen, will be called the rank of \( f \). We shall only consider non-degenerate quadratic forms; therefore the rank of a form \( (f, V) \) is the same as the rank of the lattice \( V \). By means of a chosen base, \( (e_i) \), we may express each element of \( V \) by a column matrix, e.g., \( x^t = (x_1, \ldots, x_r) \) and \( y^t = (y_1, \ldots, y_r) \), where the super-script, \( t \), denotes the transposition.

Then we have

\[ f(x, y) = x^t M y = \sum_{i, j} \alpha_{ij} x_i y_j \quad (\alpha_{ij} = \alpha_{ji}). \]

If \( (\tilde{e}_i)_{1 \leq i \leq r} \) is another base for \( V \), obtained from \( (e_i) \) by applying an invertible (\( = \) non-singular) matrix, \( P \), then the matrix, \( \tilde{M} \), of \( f \) relative to the new base is clearly given by \( \tilde{M} = P^t M P \). Thus the determinant of \( M \) and \( \tilde{M} \) differs only by the square of a unit. For a given base in \( V \), we define the determinant, \( \det f \),
of $f$ to be $\det_{\mathbb{R}} R$. Therefore $\det f$ is, apart from a factor which is the square of a unit, independent of the choice of the base in $V$.

Two quadratic forms $f = (f,V)$ and $q = (q,W)$ over the same integral domain, $A$, are said to be equivalent, in symbol $f \sim g$, if there exists an isomorphism, $u: V \rightarrow W$, of $V$ onto $W$ such that $f(x,y) = g(u(x), u(y))$. Such an isomorphism, $u$, will be referred to as an isometry (more precisely, an $(f,g)$-isometry). In particular, if $V = W$ and $f = g$, we shall call $u$ an automorph (of $f$). Suppose we are given a quadratic form $g$ on $W$ and a lattice, $V$, isomorphic to $W$. For each isomorphism, $u: V \rightarrow W$, there is a unique quadratic form, $f = u^* g$, defined on $V$, which renders $u$ an isometry.

If bases are chosen in $V$ and $W$, then $f$ and $g$ are expressed by matrices $M_f$ and $M_g$. We have, plainly, the following lemma.

**Lemma (0.3)** The quadratic forms, $f$ and $g$, are equivalent if, and only if, there exists and invertible matrix, $P$, such that $M_f = P^t M_g P$, i.e. the matrices $M_f$ and $M_g$ are congruent.

For this reason it is often convenient to use the matrical language in dealing with quadratic forms; only
in this case one has to be careful to observe what properties are invariant under a change of bases.

Let \( f_1 \) be quadratic forms defined on \( A \)-lattices \( V_1, (i = 1,2) \). A quadratic form, \( f \), defined on the direct sum, \( V_1 \oplus V_2 \), is termed the sum of \( f_1 \) and \( f_2 \), or in symbol, \( f = f_1 \oplus f_2 \), if

\[
f(x_1 \oplus x_2, y_1 \oplus y_2) = f_1(x_1, y_1) + f_2(x_2, y_2)
\]

If a base \( (e_1, \ldots, e_k; e_{k+1}, \ldots, e_s) \) is given for \( V_1 \oplus V_2 \), then the matrices, \( M, M_1 \) and \( M_2 \), of the forms \( f, f_1 \) and \( f_2 \) respectively are related by

\[
M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}
\]

If \( f \sim f_1 \oplus f_2 \), we shall say that \( f_1 \) (or \( f_2 \)) splits off from \( f \) or \( f \) decomposes into \( f_1 \) and \( f_2 \) (under the equivalence relation \( \sim \)).

A quadratic form, \( f = (f,V) \), will be called non-singular if its associated correlation \( \phi: V \rightarrow V' \) is an isomorphism. This means that the matrix, \( M \), of \( f \) is invertible or, what is the same thing, \( \det f \) is a unit in \( A \). If \( A \) is a field, then the properties of being non-singular or non-degenerate coincide. If the restriction of \( f \) on a sub-lattice, \( V_1 \), of \( V \) is non-singular, we shall say that \( f \) is non-singular on \( V_1 \). Notice that if \( f_1, f_2 \) are non-singular, so is \( f_1 \oplus f_2 \).
LEMMA (0.4) Suppose that \( f \) is a quadratic form defined on \( V_1 \oplus V_2 \), which is non-singular on \( V_1 \). Then the restriction, \( f_1 \), of \( f \) on \( V_1 \) splits off from \( f \).

In fact, with respect to a base \((e_1, \ldots, e_k; e_{k+1}, \ldots, e_r)\) of \( V_1 \oplus V_2 \), the matrix, \( M \), of \( f \) is of the form

\[
M = \begin{pmatrix} M_1 & L^t \\ L & N \end{pmatrix}, \quad M_1 = M_1^t, \quad N = N^t,
\]

where \( M_1 \) is, by assumption, invertible. Let \( P \) be the invertible matrix

\[
P = \begin{pmatrix} I & -M_1^{-1} L^t \\ 0 & I \end{pmatrix}.
\]

Then

\[
P^t MP = \begin{pmatrix} M_1 & 0 \\ 0 & -LM_1^{-1} L^t + N \end{pmatrix},
\]

which, in view of (0.3), proves the lemma.

A quadratic form \( f \) on \( V \) is termed unary, binary, ternary, \( \ldots \), \( r \)-ary, according as the rank of \( V \) is 1, 2, 3, \( \ldots \), \( r \). To assert that a quadratic form decomposes into unary forms is equivalent to saying that the matrix of the form is congruent to a diagonalized matrix.

From now on we shall assume that the integral domain, \( A \), is furthermore a local domain in the sense that there
exists a unique maximal ideal, \( \mathfrak{m} \), in \( A \). This assumption enables us to conclude the following trivial but useful properties of \( A \):

(1) The set, \( A^* \), of all the units of \( A \) coincides with \( A \setminus \mathfrak{m} \).

(ii) If \( \alpha, \beta \) are units and \( \jmath, \gamma \) are non-units in \( A \), then \( \alpha \beta \) and \( \alpha + \jmath \) are units and \( \jmath \gamma \) and \( \alpha \jmath \) are non-units.

The following theorem generalizes the classical diagonalization theorem of symmetric matrices.

**Theorem (0.5)** Every non-singular quadratic form, \( f \), over a local domain, \( A \), decomposes into unary and binary forms. If, in addition, \( 2 \in A^* \) then \( f \) decomposes into unary forms.

To prove (0.5), we chose an arbitrary base \( (e_i)_{1 \leq i \leq r} \) for the lattice \( V \) on which \( f \) is defined, and write the matrix of \( f \) as \( M = (\alpha_{ij}) \). If a diagonal entry \( \alpha_{11} \) is a unit, then the corresponding unary form splits off in virtue of (0.4). We may thus assume that no diagonal entry is in \( A^* \). Since \( \det f \) is a unit, \( \alpha_{11} \in A^* \) implies that \( \alpha_{11} \in A^* \) for some \( i \), \( 1 \leq i \leq r \); we lose nothing by assuming \( \alpha_{12} \in A^* \). But then the matrix
(0.6) \[
\begin{pmatrix}
\alpha_{11} & \alpha_{12} \\
\alpha_{21} & \alpha_{22}
\end{pmatrix}
\] \\
\[\alpha_{11} \in \mathbb{R}, \quad \alpha_{12} = \alpha_{21} \in A^*, \]

has determinant \[\alpha_{11} \alpha_{22} - \alpha_{12}^2 \in A^*\] and hence invertible. The first assertion now follows from (0.4) and a trivial induction. To prove the second assertion we need only to show that the matrix (0.6) is congruent to one whose diagonal entries are units; for then we may apply (0.4) to decompose the corresponding binary form. Indeed the assumption \[2 \in A^*\] shows that the matrix,

\[P = \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}
\]

is invertible, and

\[P^t \begin{pmatrix}
\alpha_{11} & \alpha_{12} \\
\alpha_{21} & \alpha_{22}
\end{pmatrix} P = \begin{pmatrix}
\alpha_{11} + 2\alpha_{12} + \alpha_{22} & \ast \\
\ast & \alpha_{11} - 2\alpha_{12} + \alpha_{22}
\end{pmatrix}
\]

gives the desired congruence.

Consider now a quadratic form, \(f\), defined on \(V\). If \(a \in V\) is such that \(f(a,a) = \alpha \in A^*\), then the linear map, \(u_a: V \rightarrow V\), given by

\[u_a(x) = \frac{2f(x,a)}{\alpha} a - x\]

is an involution, since \(u_a u_a = \text{identity}\). It is easy to check that \(u_a\) is an automorph leaving the element \(a\) fixed (i.e. \(u_a(a) = a\)). This automorph is also known as a reflection.
In the rest of this section we shall assume tacitly that $A$ is a local domain with a maximal ideal $\mathfrak{n} \subseteq A$ and that $2$ is a unit in $A$.

**Lemma (0.7)** Let $f = (f,V)$ be a quadratic form over $A$. If the elements $x, y \in V$ are such that $f(x,x) = f(y,y) = \alpha \in A^\times$, then there is an automorphism $u$ of $f$ which interchanges $x$ and $y$.

Indeed, the relation $f(a,a) \in A^\times$ must be satisfied by either $a = x - y$ or $a = x + y$. In the first case $u = u_a$ is the desired automorphism and in the second, $u = -u_a$ is the desired automorphism.

**Theorem (0.8)** Let $f_1, f_2$ and $h$ be quadratic forms over $A$, where $h$ is non-singular. If $f_1 + h \sim f_2 + h$ then $f_1 \sim f_2$. (Witt)

This may be regarded as the "cancellation law" for forms. Since $h$ decomposes into unary forms by (0.5), we lose nothing by assuming $h$ itself being unary in the statement of (0.8). Let the lattices on which $f_1$, $f_2$ and $h$ are defined be denoted by $V_1$, $V_2$ and $W$, respectively. Then $W$ has rank 1; let $w$ be its base. By assumption, there is an isometry $u: V_1 \oplus W \rightarrow V_2 \oplus W$. Then $(f_2 + h)(w,w) = h(w,w) = (f_1 + h)(w,w) = (f_2 + h)(u(w), u(w))$, which is a unit because $h$ is non-
singular. We apply (0.7) to obtain an automorph, \( v \), of \( f_2 + h \) such that \( v(w) = u(w) \) and \( v(u(w)) = w \). The composite map \( v : V_1 \oplus W \to V_2 \oplus W \) is an isometry which carries \( w \) into \( w \). Therefore the restriction of \( v \) on \( V_1 \) gives the required isometry to conclude \( f_1 \sim f_2 \).

For notational convenience, we shall use \( \mathbb{G}_n \) to denote the multiplicative cyclic group of order \( n \). Thus \( \mathbb{G}_n \) is actually the group \( \mathbb{Z}_n \) written multiplicatively. The multiplicative group \( A^* \), of \( A \), has a subgroup \( A^* = \{ x^2 \mid x \in A^* \} \). We shall assume, for a moment, that the ring \( A \) satisfies the additional condition

\[(0.10) \quad \frac{A^*}{A^{**}} \cong \mathbb{G}_2 \, .\]

Let the two cosets in \( \frac{A^*}{A^{**}} \) be represented by 1 and \( \xi \), \((1, \xi \in A^*)\). It follows quickly from (0.5) that every quadratic form, \( f \), over \( A \) is equivalent to one of the form

\[x_1^2 + x_2^2 + \ldots + x_k^2 + \xi x_{k+1}^2 + \ldots + \xi x_{k+m}^2 \, .\]

Let \( g \) be another quadratic form over \( A \) and

\[g \sim y_1^2 + y_2^2 + \ldots + y_r^2 + \xi y_{r+1}^2 + \ldots + \xi y_{r+s}^2 \, .\]

We may assume that \( k \leq r \). From (0.8), we see that \( f \sim g \) if, and only if

\[(0.11) \quad \xi(\tilde{x}_1^2 + \ldots + \tilde{x}_{m-s}^2) \sim (\tilde{y}_1^2 + \ldots + \tilde{y}_{r-k}^2) \, .\]

In particular, if \( A = \mathbb{R} \), the real field, then \( \xi \) may be
taken as \(-1\). The relation (0.11) holds if, and only if, 
\(m - s = r - k = 0\). This gives us *Sylvester's law of inertia*.

Recall that the determinant, \(\det f\), of a form, \(f\), is
determined up to the square of a unit. Hence for non-
singular forms over \(A\), \(\det f\) modulo \(A^{**}\) is well defined.
We shall denote by \(\text{DET } f\) the coset in \(A^{**}/A^{**}\) containing
\(\det f\). Now assume that \(A\) satisfies yet another
condition.

\[(0.12) \quad \text{the matrices } \begin{pmatrix} E & 0 \\ 0 & \phi \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ are congruent over } A.\]

Then, with the notations defined above, \(f \sim g\) if, and
only if, \(m - s = r - k\) is even; in other words, if
\(\text{DET } f = \text{DET } g\). We have shown

**THEOREM (0.13)** If \(A\) satisfies the additional
assumptions (0.10) and (0.12), then two quadratic forms,
\(f\) and \(g\), of the same rank over \(A\) are equivalent if,
and only if \(\text{DET } f = \text{DET } g\).

We have been working under the general assumption
that \(A\) is a local domain in which \(2\) is a unit. Perhaps
a few examples of such domains are now in order. In fact,
any field of characteristic different from \(2\) is such a
domain (with \(\mathfrak{m} = \{0\}\)). As examples we mention

a) \(\mathbb{Q}, \mathbb{R}, \mathbb{C}\) the fields of rational, real and
complex numbers,

b) \( \mathbb{F}(p) \) the field of \( p \)-adic numbers, where \( p \) is a (rational) prime, and
c) \( \mathbb{Z}_p \) the (finite) field of rational integers modulo \( p \), where \( p \) is an odd prime.

For local domain which are not fields we have
d) \( R(p), \mathbb{Q}(p) \) the ring of \( p \)-adic integers and the ring of rational \( p \)-adic integers (\( p \) odd prime). We may regard \( \mathbb{Q}(p) \) as \( R(p) \cap \mathbb{Q} \) which consists of all the fractions \( \frac{a}{b} \in \mathbb{Q} \) with \( b \neq 0 \) (mod \( p \)). Notice that the maximal ideal, \( \mathfrak{m} \), in \( R(p) \) is generated by the single element, \( p \in R(p) \). Indeed \( R(p) \) is a principal ideal domain in which each ideal is of the form \( p^r R(p) \). Thus it is meaningful to talk about congruence modulo \( p^r \) in \( R(p) \). We shall also make the convention that \( R(\infty) = \mathbb{F}(\infty) = \mathbb{R} \).

As one may expect, particular choice of \( A \) leads to more specific properties. We shall sample a few of them which we shall have occasion to use. If \( A = \mathbb{Z}_p \), the prime field of integers modulo \( p \), (\( p \) odd) then \( A^* \cong \mathbb{Q}_p \mathbb{Z}_p \) is of even order, and hence \( A^*/A^* \cong \mathbb{Q}_2 \). We define the Legendre symbol \( (q \mid p) \) for \( q \in A^* \) by

\[
(q \mid p) = \begin{cases} 
1 & \text{if } q \in A^* \\
-1 & \text{if } q \notin A^*
\end{cases}
\]

(0.14)
From (0.13) we have

COROLLARY (0.15) If \( A = \mathbb{Z}_p \) (p odd prime), then two quadratic forms \( f, g \) of the same rank over \( A \) are equivalent if, and only if \( (\text{det } f \mid p) = (\text{det } g \mid p) \).

To prove (0.15) we need only check that (0.12) holds in \( \mathbb{Z}_p \). Indeed, we shall prove the following lemma which clearly implies (0.12).

LEMMA (0.16) If \( \alpha \) and \( \beta \) are elements of \( \mathbb{Z}_p^\times \), then \( \alpha x^2 + \beta y^2 = 1 \) is solvable in \( \mathbb{Z}_p \).

For, if we denote by \( H \) the subset \( \{0, 1, 2, \ldots, \frac{p-1}{2}\} \) of \( \mathbb{Z}_p \), the maps \( i, j : H \to \mathbb{Z}_p \), given by

\[
i(x) = \alpha x^2, \quad j(y) = 1 - \beta y^2,
\]

are both injective. Therefore their image sets have at least one element in common. This clearly proves (0.16).

We recall that a sequence \( \{a_n\} \) of p-adic numbers converges to a limit (in \( F(p) \)) if \((a_{n+1} - a_n) \to 0\) as \( n \to \infty \), (see e.g. Van der Waerden: Algebra I, 5te Aufl. p. 255). We may also talk about convergence in lattice over \( F(p) \) or \( R(p) \) just as we do, say, in real vector spaces.

LEMMA (0.17) Let \( f = (f, V) \) be a quadratic form over \( R(p) \), not necessarily non-singular. If there
exists an element $x \in V$ such that $f(x,x) = c \pmod{p^w}$, where $c \in R(p)^\times$ and $w = 1$ or 3 according as $p$ is odd or $p = 2$, then there is $\hat{x} \in V$ such that $f(\hat{x}, \hat{x}) = c$.

Put $f(x,x) - c = p^t u$, where $t \geq 1$ if $p$ is odd and $t \geq 3$ if $p = 2$. Define

$$x_1 = x - \frac{1}{2} \cdot \frac{p^t u}{f(x,x)} \cdot x.$$ 

This is meaningful because $f(x,x)$ is invertible and, even in case where $p = 2$, the factor $\frac{1}{2}$ is admissible because $t \geq 1$. Then we have

$$f(x_1, x_1) - c = \frac{1}{4} p^{2t} u^2 f(x,x) \equiv 0 \pmod{p^{t+1}}$$

in all cases. By this means we may construct a sequence $x_1, x_2, \ldots$ of elements in a sub-lattice of rank 1 in $V$ such that $\{x_i\}$ converges to an element $\hat{x} \in V$, and $f(\hat{x}, \hat{x}) = c$. This proves the lemma.

Let $m$ be the maximal ideal in $R(p)$. Then $R(p)_{mV} = \mathbb{Z}_p$, the prime field of characteristic $p$. Thus for each $\alpha \in R(p)^\times$, $\alpha$ modulo $p$ is in $\mathbb{Z}_p^\times$. We have

COROLLARY (0.18) If $p$ is an odd prime, then

$$R(p)_{\mathbb{Z}_p^\times} \cong \mathbb{A}_2$$

and if $p = 2$, $R(2)_{\mathbb{Z}_2^\times} \cong \mathbb{A}_2 \cdot \mathbb{A}_2$.

In fact, putting $f$ to be the unary form $\alpha x^2$ in (0.17) for $\alpha \in R(p)^\times$, the case where $p$ is odd follows
immediately from the fact that \( \mathbb{Z}_p^{*}/\mathbb{Z}_p^{**} \approx \mathbb{Z}_2 \). For the case \( p = 2 \), similar argument applies and the desired result may be obtained by appealing to the properties of \( R(2) \). Using (0.16), we get

**COROLLARY (0.19)** If \( \alpha \) and \( \beta \) are elements in \( R(p)^* \) and \( p \) is odd, then \( \alpha x_1^2 + \beta x_2^2 = 1 \) has a solution in \( R(p) \).

It is of interest to study when does \( \alpha x_1^2 + \beta x_2^2 = 1 \) have a solution in other local rings. For fields \( F(2), F(3), \ldots, F(\infty) = \mathbb{R} \), this leads to the **Hilbert symbol** \((\alpha, \beta)_p\) defined as follows: for \( \alpha, \beta \in F(p)^* \), \( p = 2, 3, \ldots \) (finite primes) or \( p = \infty \),

\[
\begin{cases}
(\alpha, \beta)_p = 1 & \text{if } \alpha x_1^2 + \beta x_2^2 = 1 \text{ has a solution} \\
\text{in } F(p),
\end{cases}
\]

\[
(\alpha, \beta)_p = -1 \quad \text{otherwise.}
\]

We list with B. W. Jones the following properties of the Hilbert symbols, (see Jones: The Arithmatic Theory of Quadratic forms, Carus Monographs No. 10, p. 27).

1. \((\alpha, \beta)_\infty = 1\) unless \( \alpha \) and \( \beta \) are both negative
2. \((\alpha, \beta)_p = (\beta, \alpha)_p\)
3. \((\alpha^2, \beta\alpha^2)_p = (\alpha, \beta)_p\)
4. \((\alpha, -\alpha)_p = 1\)
5. If \( \alpha = p^a \alpha_1, \beta = p^b \beta_1 \) with \( \alpha_1, \beta_1 \in R(p)^* \), then
a. \((\alpha, \beta)_p = (-1 \mid p)^a (\alpha_1 \mid p)^b (\beta_1 \mid p)^a\),

if \(p\) is odd and \(\neq \infty\).

b. \((\alpha, \beta)_2 = (2 \mid \alpha_1)^b (2 \mid \beta_1)^a (-1)(\alpha_1 - 1)(\beta_1 - 1)/4\).

5. If \(p\) is prime to \(2\alpha\beta\), \((\alpha, \beta)_p = 1\), for \(p\) finite and \(\alpha, \beta \in R(p)\).

6. \((\alpha, \beta)_p (\alpha, \gamma)_p = (\alpha, \beta \gamma)_p\).

7. \((\alpha, \alpha)_p = (\alpha, -1)_p\).

8. \((\alpha^\circ, \beta^\circ)_p = (\alpha, \beta)_p (\alpha, -\beta)_p\).

9. If \(\beta\) is a non-square in \(F(p)\) and \(c = 1\) or \(-1\), there is for each prime \(p\) an integer \(\alpha\) such that \((\alpha, \beta)_p = c\).

10. If \(a, b \in \mathbb{Q}^*\),

\[
\prod_{p} (a, b)_p = 1
\]

where the product extends over all primes including \(\infty\).

Properties 1, 2, 3 are obvious. For property 4, notice that \(x_1 = (1 + \alpha^{-1})/2\), \(x_2 = (1 - \alpha^{-1})/2\) is a solution. Property 5' can be deduced from (0.19).

For property 5 we remark that, since \(R(p)_{x}/R(p)_{xx} \simeq G_2\) (p odd), \((\alpha_1 \mid p) = 1\) if \(\alpha_1\) is a square; otherwise \((\alpha_1 \mid p) = -1\). If \(p = 2\), then \((2 \mid \alpha_1) = (-1)(\alpha_1^2 - 1)/8\).

For property 10, we notice that the product is well defined, since all but finite factors are equal to 1. We shall omit the proofs of properties 5 through 10, (cf. Jones op. cit. pp. 28-31).
Our next goal is to define the Hasse-Minkowski symbol for a quadratic form over $\mathbb{F}(p)$ through a diagonalization of its matrix. An invariant definition of this symbol may be found in an appendix at the end of this section.

Let $\text{diag } (\alpha_1 \alpha_2 \ldots \alpha_r)$ denote the diagonalized matrix whose diagonal entries are $\alpha_1, \alpha_2, \ldots, \alpha_r$. A quadratic form, $f$, is said to be diagonal if its matrix has been diagonalized, i.e. $f$ is of the form $\alpha_1 x_1^2 + \ldots + \alpha_r x_r^2$.

**Lemma (0.20)** Let $f$ and $g$ be two diagonal forms over a field $K$. Then $f$ may be carried into $g$ through successive application of binary transformations such that at each stage the resulting form remains diagonal.

Let the matrices of $f$ and $g$ be, respectively, $M = \text{diag } (\alpha_1 \ldots \alpha_r)$ and $N = \text{diag } (\beta_1 \ldots \beta_r)$. First observe that by binary transformations we may permute $\alpha_i$'s among themselves. Our proof is by induction on $r$. The case $r = 2$ is trivial. Using theorem (0.8) and the induction hypothesis, we see that for $r > 2$ it suffices to show that $\text{diag } (\alpha_1 \ldots \alpha_r)$ may be transformed into the form $\text{diag } (\beta_1 \gamma_2 \ldots \gamma_r)$ by binary transformations such that at each stage the resulting form remains diagonal.

By assumption, there is a non-singular matrix $R = (r_{ij})$ such that $R^t M R = N$. Since $\beta_1 = \ell_{11} \alpha_1 \neq 0$ we may, by rearranging the order of $\alpha_i$'s (hence $f_{ij}$'s) if necessary, assume
that $\alpha_1 f_{11 \cdot 1}^2, \alpha_1 f_{11 \cdot 2}^2 + \alpha_2 f_{21 \cdot 2}^2, \ldots, \sum_1^r \alpha_1 f_{11 \cdot 2}^2$ are all distinct from zero. Now, the binary transformation whose matrix is of the form

$$U = \begin{pmatrix}
  f_{11} & -f_{12} & \alpha_2 & 0 \\
  f_{21} & f_{11} & \alpha_1 & 0 \\
  0 & 0 & I_{r-2}
\end{pmatrix}$$

carries $M$ into $U^T M U = \text{diag} \ ((\alpha_1 f_{11 \cdot 1}^2 + \alpha_2 f_{21 \cdot 2}^2, \alpha_3, \ldots, \alpha_r) \gamma_2$ which may then be carried by binary transformations into $\text{diag} \ ((\alpha_1 f_{11 \cdot 2}^2 + \alpha_2 f_{21 \cdot 2}^2) \alpha_3 \ldots \alpha_r) \gamma_2$). Since $\alpha_1' = \alpha_1 f_{11 \cdot 2}^2 + \alpha_2 f_{21 \cdot 2}^2 \neq 0$. Put $\alpha_2' = \alpha_3, \gamma_1' = 1$, the above construction may be repeated with $\alpha_1', \alpha_2', \gamma_1'$ in places of $\alpha_1, \alpha_2$ and $f_{11}$. The proof of (0.20) is completed.

Now let $f$ be a non-singular quadratic form over $P(p)$. We may diagonalize $f$ according to (0.5). Let the matrix of $f$ be $M_f = \text{diag} \ (\alpha_1 \ldots \alpha_r)$. We define

$$c_p(f) = c_p(\alpha_1 \ldots \alpha_r) = \prod_{1 \leq i < j} (\alpha_i, \alpha_j)_p,$$

which is either 1 or -1. We shall show that $c_p(f)$ depends only on the form, $f$, but not on the particular diagonalization $M_f$. Indeed, if $\text{diag} \ (\beta_1 \ldots \beta_r)$ gives another diagonalization of $f$ then

**LEMMA (0.22)** \[ c_p(\alpha_1 \ldots \alpha_r) = c_p(\beta_1 \ldots \beta_r). \]

For the case $r = 1$, we define $c_p(f) = 1$, and there
is nothing to prove. It is easily seen that, for \( r = 2 \), \( c_p(f) = 1 \) or \(-1\) according as \( f(x,x) = 1 \) has or has not as solution; this property is obviously independent of the diagonalization. For the general case, we observe that, by properties (2) and (3) of the Hilbert symbols,

\[ c_p(\alpha_1 \ldots \alpha_r) = c_p(\alpha_1 \alpha_2) c_p(\alpha_3 \ldots \alpha_r)(\alpha_1 \alpha_2, \alpha_3 \alpha_4 \ldots \alpha_r)_p. \]

Observe also that \( c_p(\alpha_1 \ldots \alpha_r) \) is unchanged if we permute the order of \( \alpha_i \)'s. Therefore \( c_p(\alpha_1 \ldots \alpha_r) \) is invariant under binary transformations provided that the resulting form is again diagonal. The lemma now follows from (0.20).

The symbol, \( c(f) \), which depends only on the quadratic form, \( f \), will be called the Hasse-Minkowski symbol.

**COROLLARY (0.23)** Let \( f, f_1 \) and \( f_2 \) be quadratic forms over \( F(p) \) such that \( f = f_1 + f_2 \). Then

\[ c_p(f) = c_p(f_1) c_p(f_2) (\det f_1, \det f_2)_p. \]

A quadratic form, \( f \), over \( R(p) \) can always be regarded as a form over \( F(p) \), since \( R(p) \subset F(p) \); thus the symbol \( c_p(f) \) is also defined. One deduces easily from property (5') of the Hilbert symbol that

**LEMMA (0.24)** If \( f \) is a non-singular quadratic form over \( R(p) \), \( p \neq 2 \). Then \( c_p(f) = 1 \).

By the same token we may consider \( c_p(f) \) for
\[ p = 2, 3, \ldots \infty \] if \( f \) is a quadratic form over \( \mathbb{Q} \), the field of rational numbers. Recall that if \( a, b \) are rational numbers, then \( (a,b)_p \), for \( p \) ranging over 2, 3, 5, \ldots \infty , has at most finitely many values \( \neq 1 \) and \( \prod_p (a,b)_p = 1 \). Therefore we have

**Lemma (0.25)** If \( f \) is a non-degenerate quadratic form over \( \mathbb{Q} \) then

\[
\prod_p c_p(f) = 1
\]

where \( p \) ranges over 2, 3, \ldots, \infty .
Appendix A.

Let $B$ be a commutative monoid (= semi-group) in the sense of Bourbaki (Algebra CH. I), i.e., no unit is assumed. We use $+$ to denote the monoid operation in $B$. Consider $B$ as a set of elements and let $\tilde{\mathcal{F}}(B)$ be the free abelian group generated by the elements of $B$. The group operation in $\tilde{\mathcal{F}}(B)$ will be denoted by $\cdot$. We may consider $B$ as embedded in $\tilde{\mathcal{F}}(B)$. Let $\mathcal{N}(B)$ be the subgroup of $\tilde{\mathcal{F}}(B)$ generated by the elements of the form

$$(b_1 + b_2) - b_1 - b_2 \quad (b_1, b_2 \in B)$$

and let $G(B) = \tilde{\mathcal{F}}(B)/\mathcal{N}(B)$. There is an obvious canonical map $j : B \rightarrow G(B)$ which is a monoid-homomorphism. With respect to the map $j$, $G(B)$ has the following universal property: if $h : B \rightarrow G$ is (monoid-) homomorphism of $B$ into an abelian group, $G$, then there exists a group-homomorphism $g : G(B) \rightarrow G$ such that $gj = h$; that is, the diagram is commutative.

\[
\begin{array}{ccc}
B & \xrightarrow{j} & G(B) \\
\downarrow{h} & \downarrow{g} & \downarrow{g} \\
& & G
\end{array}
\]

EXERCISE (A.1) Show that the natural map $j : B \rightarrow G(B)$ is injective if, and only if, the cancellation law holds in $B$.

EXAMPLE (A.2) Let $X$ be a topological space and let
B be the set of equivalence classes of real (resp. complex) vector bundles over X. (We do not assume X to be connected. The typical fibre may vary from (connected-) component to component.) With respect to Whitney sum, \( \oplus \), B is a commutative monoid and the group \( G(B) \) is usually denoted by \( KO(X) \) (resp. \( K(X) \)).

**Example (A.3)** Let \( A \) be an integral domain and let \( F(A) \) (resp. \( F_0(A) \)) be the set of equivalence classes of non-degenerate (resp. non-singular) quadratic forms (of various ranks) over \( A \). Then \( F(A) \) (resp. \( F_0(A) \)) is a commutative monoid with respect to the operation, \( \oplus \), defined in \( F_0 \). Notice that the cancellation law holds in \( F_0(A) \) (cf. (0.8)) if \( A \) is suitably restricted. Notice also \( F_0(A) = F(A) \) if \( A \) is a field. The group \( G(F(A)) \) (resp. \( G(F_0(A)) \)) will be denoted by \( G(A) \) (resp. \( G_0(A) \)). This example will be our main concern in this Appendix.

There is an augmentation \( \text{rk} : G(A) \to \mathbb{Z} \) defined by assigning to each class of quadratic forms the rank of a representative. Let \( f_1 = (f_1, V_1) \) and \( f_2 = (f_2, V_2) \) be quadratic forms over \( A \). We define the product, \( f_1 \otimes f_2 \), of \( f_1 \) and \( f_2 \) to be the quadratic form over \( V_1 \otimes_A V_2 \) characterized by

\[
f_1 \otimes f_2 (a_1 \otimes a_2, b_1 \otimes b_2) = f_1(a_1, b_1) f_2(a_2, b_2)
\]
where \( a_i, b_i \in V_i, i = 1,2 \). This operation, \( \otimes \), induces a ring structure on the group \( G(A) \) and the map, \( \text{rk} \), extends to a ring homomorphism, \( \text{rk} : G(A) \to \mathbb{Z} \). This ring, \( G(A) \), will be referred to as the **Grothendieck-Witt ring** (of quadratic forms over \( A \)).

**REMARK.** The groups \( \text{KO}(X) \) and \( \text{K}(X) \) are also augmentable. For simplicity, we assume \( X \) be connected. An augmentation \( \text{dim} : \text{KO}(X) \to \mathbb{Z} \) (resp. \( \text{dim} : \text{K}(X) \to \mathbb{Z} \)) may be defined by assigning to each vector bundle the real (resp. complex) dimension of its typical fibre. The tensor product of vector bundles induces a ring structure on \( \text{KO}(X) \) (resp. \( \text{K}(X) \)), known as the **Grothendieck ring** of \( X \). We shall not discuss this structure here (cf. e.g. Atiyah-Hirzebruch, Bull. AMS 65(1959) pp. 276-281).

Returning to \( G(A) \), we let \( 1 \) denote the class of quadratic forms represented by the unary form \( f = x^2 \). There is a ring-homomorphism \( \varepsilon : \mathbb{Z} \to G(A) \) defined by \( \varepsilon(1) = 1 \). Clearly \( \text{rk} \cdot \varepsilon = \text{identity} \). Let \( \hat{G}(A) \) denote the kernel of \( \text{rk} \); then \( \hat{G}(A) \) is an ideal in \( G(A) \). The short exact sequence

\[
0 \to \hat{G}(A) \to G(A) \xrightarrow{\text{rk}} \mathbb{Z} \to 0
\]

of rings and ring-homomorphisms splits since \( \text{rk} \circ \varepsilon = \text{identity on } \mathbb{Z} \). Then \( G(A) = \mathbb{Z} \oplus \hat{G}(A) \).

There is another homomorphism, \( \text{DET} : G_0(A) \to A^{\times} / A^{\times^2} \).
given by the "determinant", \( \text{DET} \), of quadratic forms. Since \( \text{DET}(f_1 + f_2) = \text{DET}(f_1) \text{DET}(f_2) \), this is a group-homomorphism. If \( A \) is such that (0.8) holds true, then we may embed \( \frac{A^*}{A^*} \) in \( G_0(A) \) by identifying each element \( a \in \frac{A^*}{A^*} \) to the class of the unary form \( ax^2 \). Observe that \( \frac{A^*}{A^*} \subset G_0(A) \) generates \( G_0(A) \) additively in view of (0.5) if \( 2 \in A^* \). If \( A = F \) is a field of characteristic \( \neq 2 \), then \( \frac{F^*}{F^*} \subset G(F) = G_0(F) \). The restriction of \( \text{DET} \) on \( \hat{G}(F) \) is already an epimorphism onto \( \frac{F^*}{F^*} \).

In fact, for \( a \in \frac{F^*}{F^*} \), \( \gamma'(a) = (a-1) \in \hat{G}(F) \) has the property \( \text{DET}(a - 1) = a \). Therefore we have the short exact sequence (of groups)

\[
(A.4) \quad 0 \rightarrow L(F) \rightarrow \hat{G}(F) \xrightarrow{\text{DET}} \frac{F^*}{F^*} \rightarrow 1 ,
\]

where \( L(F) \) is the kernel of \( \text{DET} \). The map \( \gamma' : \frac{F^*}{F^*} \rightarrow \hat{G}(F) \) given above is, in general, not a homomorphism, hence (A.4) may not split. Notice also that image of \( \gamma' \) in \( \hat{G}(F) \) generates \( \hat{G}(F) \) additively.

EXAMPLES (A.5) \( G(\mathbb{R}) \approx \mathbb{Z} + \mathbb{Z} \), \( G(\mathbb{C}) \approx \mathbb{Z} \) and \( G_0(\mathbb{Z}) \approx \mathbb{Z} + \mathbb{Z} \).

We shall only show the first assertion. If \( F = \mathbb{R} \), then \( \frac{F^*}{F^*} = \{1, -1\} \approx G_2 \). Every quadratic form may be written (uniquely) as \( \alpha^+ l + \alpha^- (-l) \). Therefore
\[ G(R) \cong \mathbb{Z} + \mathbb{Z} \]. The second assertion is trivial. We shall prove the third assertion later.

Next we propose to study \( G(F(p)) \). If \( p \) is an odd prime, then \( \frac{F^*}{F^*_p} \cong \mathbb{Q}_2 \cdot \mathbb{Q}_2 \) \( \left( F = F(p) \right) \). Indeed, from (o.18), \( \frac{R(p)^*}{R(p)^*_p} \cong \mathbb{Q}_2 \); let its elements be represented by \( \{1, \epsilon, p, p\epsilon, \} \). Then \( \frac{F^*}{F^*_p} \) is represented by \( \{1, \epsilon, p, p\epsilon, \} \cong \mathbb{Q}_2 \cdot \mathbb{Q}_2 \). Since \( (\epsilon, \epsilon)_p = 1, (\epsilon, p)_p = -(-1 \mid p) \) and \( (p, p)_p = (-1 \mid p) \), we have

\[ p \equiv 1 \pmod{4} \]

\[ (\epsilon, \epsilon)_p = 1 \]
\[ (\epsilon, p)_p = -1 \]
\[ (p, p)_p = -1 \]

**LEMMA (A.6)** For \( \alpha, \beta \in \frac{F(p)^*}{F(p)^*_p} \),

\[ (\alpha, \beta)_p = 1 \iff \alpha + \beta = 1 + \alpha \beta \text{ in } G(F(p)) \].

Indeed, \( (\alpha, \beta)_p = 1 \) means that \( \alpha x_1^2 + \beta x_2^2 = 1 \) has a solution. This is equivalent to saying that

\[
\begin{pmatrix}
\alpha & 0 \\
0 & \beta
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
1 & 0 \\
0 & \alpha \beta
\end{pmatrix}
\]

are congruent, which clearly proves the lemma. Thus we have, in \( G(F(p)) = G(F) \),

\[ p \equiv 1 \pmod{4} \quad p \equiv 3 \pmod{4} \]
\[ 2\epsilon = 2 \quad 2\epsilon = 2 \]
\[ \epsilon + p \neq 1 + \epsilon p \quad \epsilon + p \neq 1 + \epsilon p \]
\[ 2p = 2 \quad 2p \neq 2 \]
where $2 = 1 + 1$ (in $G(F)$).

In case $p \equiv 1 \pmod{4}$, we have
\[
G(F) = \mathbb{Z} \oplus \hat{G}(F)
\]
\[
= \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2
\]
\[
= 1 \quad (\varepsilon - 1) \quad (p - 1) \quad (\varepsilon - 1)(p - 1)
\]
where the element beneath a group shows a generator of that group. The last summand, $\mathbb{Z}_2$, is $L(F)$. To be precise, we must show that $(\varepsilon - 1) \neq 0$, $(p - 1) \neq 0$ and $(\varepsilon - 1)(p - 1) \neq 0$; but these follow from $\text{DET} (\varepsilon - 1) \neq 1$, $\text{DET} (p - 1) \neq 1$ and $\varepsilon + p \neq 1 + \varepsilon p$, respectively.

For the case $p \equiv 3 \pmod{4}$, we have
\[
G(F) = \mathbb{Z} \oplus \hat{G}(F)
\]
\[
= \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4
\]
\[
= 1 \quad (\varepsilon - 1) \quad (p - 1).
\]
This can be seen as follows. Property 6 of the Hilbert symbol shows that $(p, \varepsilon p)_p = (\varepsilon, p)_p (p, p) = 1$. Therefore $p + \varepsilon p = 1 + \varepsilon$, i.e. $G(F)$ is generated additively by $(\varepsilon - 1)$ and $(p - 1)$ only, and $(p - 1)$ has order 4 in $G(F)$. The element $2(p - 1)$ generates $L(F)$; therefore $L(F) \cong \mathbb{Z}_2$.

Finally, we arrive at the case where $F = F(2)$. As we know $R(2)^{\times} / R(2)^{\times} \cong \mathbb{G}_2 \cdot \mathbb{G}_2$; let it be represented by $\{1, \varepsilon, \varepsilon^{2}, \varepsilon \cdot 2^{\varepsilon} \}$, $(\varepsilon^{2} = 1, \varepsilon^{2} = 1)$. Then $F^{\times} / F^{\times} \cong \mathbb{G}_2 \cdot \mathbb{G}_2 \cdot \mathbb{G}_2$ is represented by $\{1, \varepsilon, \varepsilon^{2}, \varepsilon^{2}, 2\varepsilon^{2}, 2\varepsilon, 2^{\varepsilon}, 2^{\varepsilon} \}$; in fact, using integers modulo 8 we have $\varepsilon = 3$. 
$\mathcal{F} = 5$. Then

1. $(\epsilon, \epsilon)_2 = -1$
2. $(\epsilon, \mathcal{F})_2 = 1$
3. $(\epsilon, 2)_2 = -1$
4. $(\mathcal{F}, \mathcal{F})_2 = 1$
5. $(\mathcal{F}, 2)_2 = -1$
6. $(2, 2)_2 = 1$

Putting $x_1 = \epsilon$, $x_2 = \mathcal{F}$, $x_3 = 2$, we obtain

\[ x_1 + x_2 = 1 + x_1 x_2 \]
\[ 2x_2 = 2 \]
\[ 2x_3 = 2 \]
\[ x_1 + x_1 x_3 = 1 + x_3 \]
\[ x_1 x_2 + x_3 = 1 + x_1 x_2 x_3 \]

Using an argument similar to that used before, we get

\[ G(F(2)) = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \]
\[ = 1 \ (x_2 - 1) \ (x_3 - 1) \ (x_1 - 1) \]

and $L(F(2))$ is generated by $2(x_1 - 1)$. We leave the details to the reader.

We conclude this Appendix by giving an invariant definition of the Hasse-Minkowski symbol. For each finite prime, $p$, let $c_p^f : G(F(p)) \rightarrow L(F(p))$ be a map defined by

\[ c_p^f(f) = f - \text{DET} f \cdot 1 - \text{rk} f \cdot 1 + 1. \]

Identify $L(F(p))$ with $\mathbb{Z}_2$, then $c_p^f(f) = 0$ or 1. Define

\[ c_p(f) = (-1)^{c_p^f(f)} \]

In case $p = \infty$ (i.e. $F(p) = \mathbb{R}$), write $f = \alpha^+ \cdot 1 + \alpha^- (-1)$;
then \( \alpha^- \) is called the index of \( f \). Define

\[
c'_{\omega} (f) = (-1)^{\alpha^-} (\alpha^- - 1)/2.
\]

To see that this symbol \( c_p(f) \) coincides with the Hasse-Minkowski symbol defined in \( \mathcal{S}O \) we argue as follows.

Trivially, (write \( \text{DET} f, l \) as \( \text{DET} f \), etc.),

\[
c'_p(f + g) = f + g - \text{DET} f. \text{DET} g - \text{rk} f - \text{rk} g + 1,
\]

\[
c'_p(f) + c'_p(g) = f - \text{DET} f - \text{rk} f + 1 + g - \text{DET} g
\]
\[- \text{rk} g + 1.
\]

Therefore,

\[
(A.7) \quad c'_p (f + g) - c'_p (f) - c'_p (g) = \text{DET} f + \text{DET} g
\]
\[- \text{DET} f \text{DET} g - l.
\]

From \( (A.6) \) we have

\[
(\text{DET} f, \text{DET} g)_p = (-1)^{\text{RHS}},
\]

where \( \text{RHS} = \text{right hand side of (A.7).} \) Thus

\[
(-1)^{c'_p (f + g) - c'_p (f) - c'_p (g)} = (\text{DET} f, \text{DET} g)_p
\]

or

\[
(A.8) \quad c_p(f + g) = c_p(f) c_p(g) (\text{DET} f, \text{DET} g)_p
\]

since this newly defined \( c_p (f) \) coincides with the one defined in \( \mathcal{S}O \) for unary forms, \( f \), and since every form over \( \mathbb{F}(p) \) decomposes into unary forms, the desired result now follows from \( (A.8) \) and \( (0.5) \).

**COROLLARY (A.9)** Let \( f \) be a quadratic form over \( \mathbb{F}(p) \), \( p \) finite. Then the equivalence class of \( f \) in \( G(\mathbb{F}(p)) \) is completely determined by the \( \text{rk} f \), \( \text{DET} f \) and \( c_p(f) \).
REFERENCES


1. **Certain Arithmetical Properties of Quadratic Forms.**

Let $B$ be an integral domain which contains $A$ as a sub-domain. Then $B$ may be considered as a torsion-free module over $A$. If $V$ is an $A$-lattice, $V \otimes_A B$ is, in the natural way, a $B$-lattice whose rank over $B$ is the same as the rank of $V$ over $A$. A quadratic form $f$ defined on $V$ extends naturally into a quadratic form $f^B = (f^B, V \otimes B)$ on $V \otimes B$. This notation will be used throughout these notes.

A quadratic form over $A$ will be termed integral, rational or real according as $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. Let $f$ be a real quadratic form whose matrix has been diagonalized according to (0.5). Let $\alpha^+$ (resp. $\alpha^-$) denote the number of positive (resp. negative) diagonal entries. Then the integers $\alpha^+$ and $\alpha^-$ are independent of the diagonalization in virtue of Sylvester's law of inertia, which we have shown in \(\hat{1}\). From what we just said, we may regard $\alpha^+$ and $\alpha^-$ as being defined for integral and rational forms also. Since we only consider non-degenerate quadratic forms, $\alpha^+ + \alpha^- = \text{rk}(f)$, the rank of $f$. Define the signature, $\tau(f)$, of the form $f$ to be the integer $\alpha^+ - \alpha^-$. Notice that $\alpha^+$ and $\alpha^-$, and hence the signature, are also defined for forms over $Q(p) = \mathbb{Q} \cap \mathbb{R}(p)$, the ring of rational $p$-adic integers.
A quadratic form, $f$, over $\mathbb{Z}$ or $\mathbb{R}(2)$ is said to be **even** if $f(x,x) \equiv 0 \pmod{2}$ for every $x \in V$, where $V$ is the lattice on which $f$ is defined; otherwise $f$ is termed odd. Since

$$f(x + y, x + y) = f(x, x) + f(y, y) + 2f(x, y),$$

$f$ is even if, and only if, the diagonal entries of its matrix are all even, where an element $t \in \mathbb{R}(2)$ is called even or odd according as $t \equiv 0 \pmod{2}$ or not.

Consider a quadratic form, $f = (f, V)$, over $\mathbb{Z}$ or $\mathbb{R}(2)$ with odd determinant (this property clearly independent of the base chosen in $V$). There exists an element $w \in V$, in general not unique, satisfying the relation

$$f(x, x) \equiv f(x, w) \pmod{2}$$

for each $x \in V$. For if $x = \sum_{i} x_{i} e_{i}$ and $w = \sum_{i} w_{i} e_{i}$, where $e_{1}, ..., e_{r}$ constitute a base of $V$ and $w_{1}, x_{1}$ are elements in $\mathbb{Z}$ or $\mathbb{R}(2)$ as the case may be, then

$$f(x, x) = f(\sum_{i} x_{i} e_{i}, \sum_{i} x_{i} e_{i})$$

$$= \sum_{i} f(e_{i}, e_{i}) x_{i}^{2} \pmod{2}.$$  

On the other hand,

$$f(x, w) = \sum_{i,j} f(e_{i}, e_{j}) x_{i} w_{j}.$$  

The relation (1.2) is equivalent to

$$f(e_{1}, e_{1}) \equiv \sum_{j} f(e_{1}, e_{j}) w_{j} \pmod{2}.$$  

Since $\det f \not\equiv 0 \pmod{2}$, we may solve for the coefficients
$w_j$, $j = 1, \ldots, r$. Such an element, $w$, will be called a **characteristic element** of $f$. If $\hat{w}$ is another characteristic element of $f$, then $f(x, \hat{w} - w) \equiv 0 \pmod{2}$ for every $x \in V$. It follows from the assumption $\det f \not\equiv 0 \pmod{2}$ that

\[(1.3) \quad \hat{w} = w + 2z\]

for some $z \in V$. Conversely, any element, $w$, satisfying (1.3) is a characteristic element whenever $w$ is one. Now

$f(\hat{w}, \hat{w}) = f(w, w) + 4f(w, z) + 4f(z, z)$.

We conclude from (1.2) that

\[(1.4) \quad f(w, w) \equiv f(\hat{w}, \hat{w}) \pmod{8}\]

It is scarcely necessary to remark that, in $R(2)$, mod 8 means modulo the ideal $2^3 R(2)$. Since the integers, $\mathbb{Z}$, form a sub-ring of $R(p)$ for $p = 2, 3, \ldots, \infty$; we may, in particular, consider an integral form as a form over $R(2)$.

**THEOREM** (1.5) Let $f$ be a quadratic form over $R(2)$ with odd determinant. Then

\[(1.6) \quad f(w, w) - r - \det f + l \equiv 0 \pmod{4}\]

where $r = \text{rk } f$ is the rank of $f$.

In view of (1.4), the relation (1.6) is meaningful if $\det f$ modulo 4 is well-defined; but this is clear since $\det f$ is determined up to the square of a unit and we are
working in \( \mathbb{R}(2) \). One additional remark before we proceed to prove (1.5): throughout these notes the equality sign, \( = \), will also stand for the phrase "is canonically isomorphic to".

Let \( f = (f, V) \) be the given quadratic form. If \( r = 1 \), then \( V = \mathbb{R}(2) \) and \( f(x, y) = axy \), is odd, where \( a = \det f \) (with respect to the canonical base). Thus \( w = 1 \) is a characteristic element of \( f \) and (1.6) is clearly satisfied if \( r = 2 \) then \( V = \mathbb{R}(2) \oplus \mathbb{R}(2) \). If furthermore \( f \) is an even form, the matrix of \( f \) relative to the canonical base may be written as

\[
\begin{pmatrix}
2a & c \\
c & 2b
\end{pmatrix}
\]

where \( c \) is odd because \( \det f \) is odd. Therefore \( c^2 \equiv 1 \pmod{4} \). Now, \( w = 0 \) is a characteristic element of \( f \), so we have \( f(w, w) - r - \det f + 1 = 0 - 2 - 4ab + c^2 + 1 \equiv 0 \pmod{4} \). Thus (1.5) is proved for unary and binary even forms.

Assume that \( f = f_1 + f_2 \) is a decomposition of \( f \), where \( f_1 \) is defined on \( V_1 \), \( i = 1, 2 \), and \( V_1 \oplus V_2 = V \). Obviously \( f_1 \) has odd determinant and \( r_1 + r_2 = r \) where \( r_1 \) is the rank of \( f_1 \) (\( i = 1, 2 \)). Suppose that, by induction, the theorem has been proven for quadratic forms of rank \( \leq \max(r_1, r_2) \). Let \( w_1 \) be a characteristic element
of \( f_1 (i = 1,2) \). Then \( w = w_1 \oplus w_2 \) is a characteristic element of \( f \). Since \( \det f_1 \) is odd for \( i = 1,2 \), we have
\[
(\det f_1 - 1)(\det f_2 - 1) = \det f_1 \det f_2 - \det f_1 - \det f_2 + 1 \equiv 0 \pmod{4}.
\]
On the other hand,
\[
f(w,w) - r - \det f + 1
= f_1^{-1}f_2(w_1 \oplus w_2, w_1 \oplus w_2) - r_1 - r_2 - \det f_1 \det f_2 + 1
= f_1(w_1, w_1) + f_2(w_2, w_2) - r_1 - r_2 - \det f_1 \det f_2 + 1
\equiv -\det f_1 \det f_2 + \det f_1 + \det f_2 - 1 \pmod{4},
\]
by induction hypothesis. Therefore
\[
f(w,w) - r + \det f + 1 \equiv 0 \pmod{4}.
\]
To complete the proof we need only show that every quadratic form of odd determinant may be written as a sum of unary and binary even forms with odd determinants. A slight modification of the proof of (0.5) will prove the last statement and hence the theorem.

Now, for an element \( a \in R(2) \), the exponent \((-1)^a\) may be defined by stipulating
\[
(-1)^a = \begin{cases} 
1, & \text{if } a \text{ is even} \\
-1, & \text{if } a \text{ is odd}.
\end{cases}
\]
We then define
\[
\bar{e}_2(f) = (-1)^a \frac{f(w,w) - r - \det f + 1}{4}.
\]
Recall that for a quadratic form, \( f \), over \( R(2) \) the
Hasse-Minkowski symbol $c_2(f)$ is also defined. We have

$$\text{THEOREM (1.8)} \quad \tilde{c}_2(f) = c_2(f).$$

Indeed, if $f_1$ and $f_2$ (hence $f = f_1 + f_2$) are forms over $R(2)$ with odd determinant, then in virtue of (1.7) and property (5b) of the Hilbert symbols we have

$$\tilde{c}_2(f) = \tilde{c}_2(f_1) \tilde{c}_2(f_2) \left(\det f_1, \det f_2\right)_2.$$ 

Comparing this with (0.23) we see that $\tilde{c}$ and $c$ behave the same way on sum of forms. It follows that the proof of (1.8) will be completed if we can prove the statement for unary and binary even forms. With the notations used in the proof of (1.5) we see that if $f$ is unary, $R(2) = V$, $w = 1$ and $\det f = a \in R(2)^*$ then $\tilde{c}_2(f) = 1 = c_2(f)$. If $f$ is a binary even form, $V = R(2) \oplus R(2), \, w = 0$, then

$$c_2(f) = \frac{-1 - 4ab + c^2}{4} = (-1)^{ab}$$

because $c^2 - 1 \equiv 0 \pmod{8}$. We consider two cases:

**case 1.** $a = b = 0$. In this case the matrix of $f$ is

$$\begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$$

which is congruent over $F(2)$ to $\text{diag}(\alpha, -\alpha)$. Therefore $c(f) = 1$. Trivially, $\tilde{c}_2(f) = 1$; hence $c_2(f) = \tilde{c}_2(f)$.

**case 2.** At least one of $a$ and $b$ is not zero, let it be $a$. We split $2a$ off from $f$ to obtain
\[ c_2(f) = (2a, \frac{4ab - c^2}{2a})_2 \]
\[ = (2a, \frac{4ab - c^2}{2a})_2 (2a, -2a)_2 \]
\[ = (2a, c^2 - 4ab)_2 \]

Recall that \((2|\alpha) = 1\) or \(-1\) according as \(\alpha \equiv 1, 7 \pmod{8}\) or \(\alpha \equiv 3, 5 \pmod{8}\). It is now a routine matter to check that \(\tilde{c}_2(f) = c_2(f)\) (use property (5b) of the Hilbert symbols). This proves case 2 and hence the theorem.

Now let \(f\) be a quadratic form over \(\mathbb{Q}(2) = \mathbb{R}(2) \cap \mathbb{Q}\). Then the signature, \(\tau = \tau(f)\), is defined. We have

**THEOREM (1.9)** For non-singular quadratic forms, \(f\), over \(\mathbb{Q}(2)\),

\[ c_2(f) c_\infty(f) = (-1)^{\frac{f(w, w) - \tau - \det f + \text{sgn} \det f}{4}} \]

Let \(\alpha^+\) and \(\alpha^-\) be defined as at the beginning of this section. Then \(\tau = \alpha^+ - \alpha^-\), \(r = \alpha^+ + \alpha^-\) and, obviously, \(\text{sgn} \det f = (-1)^{\alpha^-}\). In view of Theorem (1.8), we consider

\[ \frac{f(w, w) - \tau - \det f + \text{sgn} \det f}{4} - \frac{f(w, w) - r - \det f + 1}{4} \]

\[ = \frac{-\tau + r + \text{sgn} \det f - 1}{4} \]

\[ = \frac{2\alpha^- + (-1)^{\alpha^-} - 1}{4} = \frac{\alpha^- (\alpha^- - 1)}{2} \pmod{2}. \]

It is easy to compute \(c_\infty(f)\) directly from the definition.
and property (1) of the Hilbert symbols; indeed
\[ c_\infty(f) = (-1) \frac{a^-(a^- - 1)}{2} \]
This completes the proof of the theorem.

Since the only units in \( \mathbb{Z} \) are 1 and -1, the determinant \( \det f \) of an integral form \( f \), is well-defined. An integral form is called unimodular (= non-singular over \( \mathbb{Z} \)) if its determinant is \( \pm 1 \); in this case \( \det f - \text{sgn} f = 0 \). Furthermore, for such forms \( f \), \( c_p(f) = 1 \) if \( p \) is an odd prime, in virtue of property \( (5') \) of the Hilbert symbols. From Lemma (0.25) we deduce that \( c_2(f) c_\infty(f) = 1 \). This together with (1.9) proves the following

**COROLLARY (1.10)** If \( f \) is an integral unimodular quadratic form then
\[ f(w, w) - \tau \equiv 0 \pmod{8}. \]

In particular, if \( f \) is, in addition, an even form, then \( w = 0 \) is a characteristic element. Thus we have

**COROLLARY (1.11)** If \( f \) is an integral, unimodular, even quadratic form then
\[ \tau(f) \equiv 0 \pmod{8}. \]

We remark that this last result is the best possible in the sense that there actually exists a quadratic form satisfying the hypothesis with \( \tau = 8 \). Indeed, let \( E_8 \)
denote the following graph (= 1-dimensional simplicial complex, cf. §4)

Construct a matrix $M = (\kappa_{ij})$ with integral entries according to the following formulae

$\kappa_{ii} = 2$, for $i = 1, 2, \ldots, 8,$

$\kappa_{ij} = 1$, if the vertices $v_i$ and $v_j$ are joined by an edge (= 1-simplex) in $E_8$,

$\kappa_{ij} = 0$, otherwise.

Thus

$$M = \begin{pmatrix}
2 & 1 & & & & & & \\
1 & 2 & 1 & & & & & \\
1 & 2 & 1 & 0 & & & & \\
1 & 2 & 1 & & 0 & & & \\
1 & 2 & 1 & & & 0 & & \\
0 & & 1 & 2 & 0 & & & \\
0 & & & & & 1 & 0 & 2 \\
& & & & & & & 
\end{pmatrix}.$$

The corresponding even quadratic form is easily seen to be unimodular with signature $\tau = 8$.

We call attention to this method of constructing integral forms from a graph. Similar constructions will be used later. Before a more general definition is given we shall refer to the quadratic form mentioned above as "the" quadratic form associated to $E_8$. 
2. **The Quadratic form of a 4k-dimensional Manifold.**

Unless otherwise specified, by a manifold we mean a connected, compact, orientable, differentiable manifold with or without boundary, together with a given orientation. Thus all manifolds under consideration are oriented. An n-dimensional manifold is also called an n-manifold.

Let $M$ be a 4k-manifold. The homology group $H_{2k}(M; \mathbb{Z})$ is finitely generated. Therefore

$V = H_{2k}(M; \mathbb{Z})/(\text{Torsion})$ is a lattice over $\mathbb{Z}$. There is a pairing,

$$S : V \times V \rightarrow \mathbb{Z},$$

defined by the intersection number of cycles. Precisely, if $a, b$ are elements in $H_{2k}(M; \mathbb{Z})$, represented respectively by cycles $z_1, z_2$ in $Z_{2k}(M; \mathbb{Z})$, we define $\tilde{S}(a,b) = z_1 \cdot z_2$, the intersection number of $z_1$ and $z_2$. This pairing is clearly well-defined, bilinear and, since $2k$ is an even number, symmetric. Observe that $\tilde{S}(a,b)$ vanishes if $a$ or $b$ is a torsion element and hence it induces a quadratic form $S = (S,V)$, on $V$ into the integers. Henceforth, this quadratic form will be written as $S_M$, and will be called the **quadratic form** of $M$. We shall also write $\tau(M)$ for $\tau(S_M)$, the signature of $S_M$.

The Poincaré duality for an n-manifold, $M$, may be expressed by an isomorphism
\[ H_1(M; \mathbb{Z}) \cong H^{n-1}(M, \partial M; \mathbb{Z}) \]

for each \( 1 \leq i \leq n \). In particular, for \( n = 4k \), \( i = 2k \), we have

\[ H_{2k}(M; \mathbb{Z}) \cong H^{2k}(M, \partial M; \mathbb{Z}). \]

Denote the elements in \( H^{2k}(M, \partial M; \mathbb{Z}) \) which correspond to \( a, b, \ldots \), in \( H_{2k}(M; \mathbb{Z}) \) under this isomorphism, by \( \alpha, \beta, \ldots \), respectively. Since

\[ \alpha \cup \beta \in H^{4k}(M, \partial M; \mathbb{Z}) = \mathbb{Z}, \]

(where the equality sign, as remarked before, means "is canonically isomorphic to"), we may consider \( \alpha \cup \beta \) as an integer, which is precisely the intersection number of \( a \) and \( b \) (see e.g. [H-W], p. 156). In this sense we may regard \( S_M \) as a pairing defined on \( H^{2k}(M, \partial M; \mathbb{Z})/(\text{Torsion}) \) into \( \mathbb{Z} \) by the cup-product.

**Exercise (2.1)** Show that if \( M \) is a 4k-dimensional unbounded manifold (i.e. \( \partial M = \emptyset \)) then \( S_M \) is unimodular (see e.g. Milnor [1]).

Our main concern, at least for a while, will be 4-manifolds. Many results stated below have higher dimensional analogues; but this does not concern us at this moment.

Let \( M \) be an unbounded 4-manifold, and let \( H^*(M; \mathbb{Z}_2) \) be its cohomology ring with coefficients in \( \mathbb{Z}_2 \). The quadratic form of \( M \) over \( \mathbb{Z}_2 \) is again given by the cup-product: \( S(x, y) = x \cup y \). In this
case there exists a unique characteristic element \( w_2 = w_2(M) \) in \( H^2(M; \mathbb{Z}_2) \) satisfying \( x \cup x = x \cup w_2 \), for each \( x \in H^2(M; \mathbb{Z}_2) \). We remark that \( w_2 \) is actually the middle Stiefel-Whitney class of \( M \) since \( M \) is oriented.

The following theorems are known. (Rohlin [2], Borel-Hirzebruch [3], see also Kervaire-Milnor [1] and [2]).

**Theorem (2.2)** If \( M \) is an unbounded 4-manifold with \( w_2 = 0 \), then \( \tau(M) \equiv 0 \pmod{16} \). (Rohlin)

We remind the reader that all manifolds under consideration are supposed to be differentiable. We do not know whether (2.2) holds in general for oriented topological 4-manifolds. (Notice that the statement of (2.2) still makes sense in this case.)

Let \( \pi : H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{Z}_2) \) be the reduction modulo 2, i.e. the homomorphism induced by the coefficient epimorphism \( \mathbb{Z} \to \mathbb{Z}_2 \). We have

**Theorem (2.3)** If \( d \in H^2(M; \mathbb{Z}) \) is such that \( \pi d = w_2 \), then \( S_M(d,d) \equiv \tau(M) \pmod{8} \), where \( S_M \) is the integral quadratic form of \( M \). (Borel-Hirzebruch)

The assumption of (2.3) implies that \( d \) is a characteristic element of \( S_M \), (because the cup-product commutes with the homomorphism \( \pi \)). Hence (2.3) follows immediately from (1.1a) and (2.1). A weakened
version of (2.2), i.e., \( \tau(M) \equiv 0 \pmod{8} \), can then be deduced from (2.3).

The short exact sequence \( 0 \to \mathbb{Z}^2 \to \mathbb{Z} \to \mathbb{Z}_2 \to 0 \) of coefficients induces an exact sequence

\[
\cdots \to H^2(M; \mathbb{Z}) \xrightarrow{2} H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{Z}_2) \to \cdots
\]

in homology, where the number 2 above an arrow means multiplication by 2. If \( M \) has no 2-torsion (i.e., the 2-component of the torsion subgroup of \( H^w(M; \mathbb{Z}) \) is zero), then (2.4) gives rise to a short exact sequence

\[
0 \to H^2(M; \mathbb{Z}) \xrightarrow{2} H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{Z}_2) \to 0.
\]

Therefore,

\[
H^2(M; \mathbb{Z}_2) \cong \frac{H^2(M; \mathbb{Z})}{2 \cdot H^2(M; \mathbb{Z})}
\]

Thus \( w_2 \) corresponds, under the above isomorphism, to the residue class containing the characteristic elements of \( S_M \). We conclude that, assuming \( M \) has no 2-torsion, \( w_2 = 0 \) if, and only if, the quadratic form \( S_M \) is even. We may thus state

**Theorem (2.2a)** If \( M \) is a 4-manifold without 2-torsion and \( S_M \) is even, then \( \tau(M) \equiv 0 \pmod{16} \).

Hence the quadratic form associated to \( E_8 \) mentioned in §1 cannot occur as the quadratic form of a 4-manifold without 2-torsion. The following
theorem, due to Kervaire and Milnor [2], sharpens the results of (2.2) and (2.3).

**THEOREM** (2.5) In the statement of (2.3) if the dual class of \( d \) is represented by a differentiably imbedded 2-sphere in \( M \), then \( S(d,d) \equiv \tau(M) \pmod{16} \).

We hope to include later a proof of this theorem and some examples.

3. **An application of Rohlin's theorem, \( \mu \)-invariants.**

Let \( G \) be an abelian group. By a \( G \)-homology \( k \)-sphere, \( X \), we mean an unbounded manifold whose homology groups over \( G \) are isomorphic to those of a \( k \)-sphere; that is, \( H_i(X;G) \cong G \), for \( i = 0, k \), and \( H_i(X;G) = 0 \), otherwise. Note that if \( G \neq 0 \), this implies that \( X \) is \( k \)-dimensional.

We are interested in a particular class, \( \gamma_3 \), of \( \mathbb{Z}_2 \)-homology 3-spheres. A \( \mathbb{Z}_2 \)-homology 3-sphere, \( X \), is an element of \( \gamma_3 \) if, and only if, \( X \) bounds a 4-manifold \( Y(\text{i.e. } \gamma Y = X) \) satisfying the conditions

(3.1) a) \( H_1(Y;\mathbb{Z}) \) has no 2-torsion, and

b) \( S_Y \) is an even quadratic form.

We remark that from cobordism theory, every 3-manifold bounds. Here we merely impose an additional condition, (3.1), to the manifolds they bound. For each \( X \in \gamma_3 \), we define

\[
\mu(X) = \frac{T(Y)}{16} \in \mathbb{Q}/\mathbb{Z}
\]
(the number $-\tau(Y)/16$ reduced modulo 1). It is understood that $Y$ satisfies (3.1).

**THEOREM (3.2)** For $X \in \partial Y$, $\kappa(X)$ is invariant under orientation-preserving diffeomorphisms.

For this reason we shall call $\kappa(X)$ the $\kappa$--invariant of $X$. This is a special case of the $\kappa$--invariants studied by Bells and Kuiper [1]. Clearly, to prove (3.2) it suffices to show that $\kappa(X)$ is independent of the choices of $Y$ (provided that $Y$ satisfies (3.1) and $\partial Y = X$, of course). Let $Y_1$ and $Y_2$ be two such 4-manifolds with $\partial Y_1 = \partial Y_2 = X$. Without loss of generality, we may assume that $Y_1 \setminus X$ and $Y_2 \setminus X$ have no common points. Let $-Y_2$ denote the manifold obtained from $Y_2$ by reversing its orientation. Then $M = Y_1 \cup (-Y_2)$ is a topological manifold, oriented in accordance with $Y_1$ and with $-Y_2$. In other terms, $M$ is constructed from $Y_1$ and $-Y_2$ by "pasting" their boundaries together according to the identity map on $X$. Notice that $M$ has, after smoothing if necessary, a differentiable structure compatible with those on $Y_1$ and on $-Y_2$. Thus $M$ is a manifold in our sense. We refer the reader to Milnor [2], [3] for details about pasting and smoothing.

**LEMMA (3.3)** The manifold, $M$, obtained above has the following properties:
1) \( M \) has no 2-torsion,

2) \( S_M \) is even, and

3) \( S_{Y_1}^Q + (-S_{Y_2}^Q) = S_M^Q \);

where \( S^Q \) is defined as in \( \phi 1 \).

Before we proceed to prove (3.3), observe that the lemma is sufficient to conclude (3.2). For property 3) implies \( \tau(M) = \tau(Y_1) - \tau(Y_2) \), and properties 1) and 2) together with Rohlin's theorem (2.2a), imply that \( \tau(M) \equiv 0 \pmod{16} \); Theorem (3.2) then follows. To prove (3.3) we need

**LEMMA (3.4)** If \( X \) is a \( \mathbb{Z}_2 \)-homology k-sphere then

a) \( H_i(X; \mathbb{Z}) \) is a torsion group of odd order for \( i \neq 0, k \),

b) \( X \) is a \( \mathbb{Q} \)-homology sphere, and

c) \( X \) is an \( R(2) \)-homology sphere.

Consider the exact sequence of homology groups

\[
\ldots \rightarrow H_{i+1}(X; \mathbb{Z}_2) \rightarrow H_i(X; \mathbb{Z}) \rightarrow H_i(X; \mathbb{Z}_2) \rightarrow \ldots
\]

induced by the coefficient exact sequence

\[
0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0.
\]

If \( i \neq 0, k \), then

\[
H_i(X; \mathbb{Z}) \rightarrow H_i(X; \mathbb{Z}_2)
\]

is an epimorphism. As the groups involved are finitely generated, we conclude that

\( H_i(X; \mathbb{Z}) \) is a torsion group of odd order. Since b) is an obvious consequence of a), it remains to prove c).

Using the exact sequence \( 0 \rightarrow R(2) \rightarrow R(2) \rightarrow \mathbb{Z}_2 \rightarrow 0 \), we may replace \( \mathbb{Z} \) by \( R(2) \) in (3.5) to obtain
\[ \cdots \to H_{1+1}(X; \mathbb{Z}_2) \to H_1(X; \mathbb{Z}_2) \to H_1(X; \mathbb{R}(2)) \to H_1(X; \mathbb{R}(2)) \to \cdots \] Recall that \( \mathbb{R}(2) \) is a PID (principal ideal domain) and \( H_1(X; \mathbb{R}(2)) \) is finitely generated. Hence

\[(3.6) \quad H_1(X; \mathbb{R}(2)) \cong \bigoplus_{k=1}^{m} \mathbb{R}(2)/\alpha_k^2,\]

where \( \alpha_k \)'s are ideals of \( \mathbb{R}(2) \) such that
\[\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}_m \neq \mathbb{R}(2) \quad (\text{cf. [B-a], Ch. 7, \( \S \) 4, Theorem 2}).\] The only ideals in \( \mathbb{R}(2) \) are the zero ideal and those generated by \( 2^j \); therefore each summand in (3.6) is of the form \( \mathbb{R}(2) \) or \( \mathbb{R}(2)/2^j \). It follows that if \( H_1(X; \mathbb{R}(2)) \to H_1(X; \mathbb{R}(2)) \) is an epimorphism, then \( H_1(X; \mathbb{R}(2)) = 0 \). This proves c) and hence (3.4).

Returning to (3.3), we consider \( X, Y_1, Y_2 \), and \( M \) as defined there and apply the Mayer-Vietoris sequence to obtain

\[(3.7) \quad \cdots \to H_1(X) \to H_1(Y_1) \oplus H_1(Y_2) \to H_1(M) \to H_{1-1}(X) \to \cdots,\]

where the coefficient group is not specified. From (3.4) we know that \( X \) is a \( \mathbb{Q} \)-homology 3-sphere.

Therefore \( H_2(M; \mathbb{Q}) \) has an injective representation.

\[(3.8) \quad 0 \to H_2(Y_1; \mathbb{Q}) \oplus H_2(Y_2; \mathbb{Q}) \to H_2(M; \mathbb{Q}) \to 0.\]
Taking orientations into account, it is not hard to see that \( S_{y_1}^Q + (-S_{y_2}^Q) = S_M^Q \). This proves 3) of (3.3).

Next, we take integral coefficients in (3.7) to obtain
\[
\ldots \rightarrow H_1(X; \mathbb{Z}) \rightarrow H_1(y_1; \mathbb{Z}) \oplus H_1(y_2; \mathbb{Z}) \rightarrow H_1(M; \mathbb{Z}) \\
\rightarrow \tilde{H}_0(X; \mathbb{Z})
\]
where \( \tilde{H}_0(X; \mathbb{Z}) = 0 \) is the reduced homology group. Since \( y_1 \) and \( y_2 \) have no 2-torsion and since \( H_1(M; \mathbb{Z}) \) is a torsion group (by (3.4a)), it follows that \( H_1(M; \mathbb{Z}) \) has no 2-component. Property 1) of (3.3) now follows from Poincaré duality (of homology groups, see e.g. [S-T] p. 245, Satz II).

Finally, we take \( R(2) \) as coefficient group in (3.7) and apply (3.4) to obtain an injective representation
\[
0 \rightarrow H_2(y_1; R(2)) \oplus H_2(y_2; R(2)) \rightarrow H_2(M; R(2)) \rightarrow 0
\]
just as in (3.8). Since \( \text{Tor}(\text{odd torsion, } R(2)) = 0 \), then by the universal coefficient theorem,
\[
H_2(y_1; R(2)) = H_2(y_1; \mathbb{Z}) \otimes R(2), \quad i = 1, 2, \text{ and}
\]
\[
H_2(M; R(2)) = H_2(M; \mathbb{Z}) \otimes R(2). \quad \text{Recall that } S_{y_1}^{R(2)}
\]
is defined on \( \frac{H_2(y_i; \mathbb{Z})}{\text{Torsion}} \otimes R(2) \approx \frac{H_2(y_i; \mathbb{Z})}{\text{Torsion}} \otimes R(2), \quad i = 1, 2. \) Thus
\[
S_M^{R(2)} = S_{y_1}^{R(2)} + (-S_{y_2}^{R(2)}). \]
The last two terms are even by assumption; hence $S^R(2)$ is also even. The last statement implies that $S_M$ itself is even, which completes the proof of (3.3).

As an example we shall apply the invariant $\mu(X)$ to lens spaces. Let the standard 3-sphere, $S^3$, be represented in the complex 2-space, $\mathbb{C}^2$, by

$$S^3 = \{(z_1,z_2) \mid z_1\overline{z}_1 + z_2\overline{z}_2 = 1\}.$$ Then $S^3$, as submanifold of $\mathbb{C}^2$, inherits a canonical orientation.

Let $n$ and $q$ be integers, $0 < q < n$, such that $(n,q) = 1$, (i.e., $n$ and $q$ are co-prime). Define an operation of the group $\mathbb{Z}_n$ on $S^3$ by the formula

$$\gamma, (z_1,z_2) = (e^{2\pi i/n} z_1, e^{2\pi i q/n} z_2)$$

for each $\gamma \in \mathbb{Z}_n$. Then $\mathbb{Z}_n$ acts freely and differentiably on $S^3$. The quotient space (space of orbits) $S^3/\mathbb{Z}_n = L(n,q)$ is, by definition, the lens space of type $(n,q)$, which inherits an orientation and a differentiable structure from $S^3$. Therefore $L(n,q)$ is a manifold in our sense. Since $L(n,q)$ has $S^3$ as its universal covering space,

$$\pi_1(L(n,q)) = \mathbb{Z}_n = H_1(L(n,q); \mathbb{Z}).$$

Thus the Betti numbers of $L(n,q)$ are $1,0,0,1$ and there is a torsion coefficient, $n$, at dimension 1. If $n$ is odd, $L(n,q)$ is a $\mathbb{Z}_2$-homology sphere.
THEOREM (3.10) For each odd integer \( n > 0 \),
\[ L(n,q) \in \mathcal{C}_3. \]

A proof of this theorem can be found in \( \rho \).

Granting this, we proceed to study the \( \alpha \)-invariant of such lens spaces.

LEMMA (3.11) With respect to the canonical orientations, \( L(n,q) = -L(n,n-q) \).

In fact, the orientation reversing differentiable involution, \( \alpha : (z_1,z_2) \to (z_1,z_2) \), on \( S^3 \) carries \( L(n,n-q) \) onto \( -L(n,q) \). Explicitly, for \( \gamma \in \mathbb{Z} \),
\[ (e^{\frac{2\pi i}{n}}z_1, e^{\frac{2\pi i}{n}}z_2) = (e^{\frac{2\pi i}{n}}z_1, \]
\[ e^{\frac{2\pi i}{n}}(z_1, z_2) = \alpha(e^{\frac{2\pi i}{n}}z_1, e^{\frac{2\pi i}{n}}z_2). \]
Thus, to study the \( \alpha \)-invariants of \( L(n,q) \), we may restrict our attention to the case where \( q \) takes even values. Recall that a fraction \( n/q \) can be expanded into (finite) continued fractions
\[ \frac{n}{q} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \cdots - \frac{1}{b_s}}}. \]

\[ = [b_1, b_2, b_3, \ldots, b_s] \quad \text{(notation)} , \]
where \( b_1 \) is an integer with \( |b_1| > 2, i = 1, 2, \ldots, s \).
In case where \( n \) is odd and \( q \) is even, there is a unique expansion \( n/q = [b_1, b_2, \ldots, b_s] \) such that
each \( b_1 \) is even (the proof is elementary). Denote by \( p^+ = p^+(n,q) \) (resp. \( p^- = p^-(n,q) \)) the number of positive (resp. negative) \( b_1 \)'s in this unique expansion of \( n/q \). We have

\[
\mu(L(n,q)) = \frac{p^+ - p^-}{16} \in \mathbb{Q}/\mathbb{Z}.
\]

A proof of this fact will be given later. As examples we have

\[
\begin{align*}
7/6 &= [2,2,2,2,2,2] & p^+ &= 6, p^- = 0, \\
7/2 &= [4,2] & p^+ &= 2, p^- = 0.
\end{align*}
\]

Therefore,

\[
\begin{align*}
\mu(L(7,6)) &= 3/8, \\
\mu(L(7,2)) &= 1/8, \\
\mu(L(7,1)) &= -\mu(L(7,6)) = -3/8 = 5/8 \pmod{1}.
\end{align*}
\]

We recall that (J. H. C. Whitehead [1]),

THEOREM (3.13) Two lens spaces, \( L(n,q) \) and \( L(n,q') \), have the same homotopy type if, and only if, \( q q' \) or \(-q q'\) is a quadratic residue modulo \( n \).

Thus \( L(7,1) \) and \( L(7,2) \) have the same homotopy type; but they are not diffeomorphic because \( \mu(L(7,1)) \) differs from \( \mu(L(7,2)) \) and from \(-\mu(L(7,2)) \).

Now let \( X_1 \) and \( X_2 \) be unbounded \( n \)-manifolds. We say that \( X_1 \) and \( X_2 \) are \textit{h-cobordant} (\( \varepsilon \)-\textit{J-equivalent}) if, and only if, there exists an \((n+1)\)-manifold, \( W \), such that

1) \( W = X_1 \cup -X_2 \) (disjoint union) and,

2) both \( X_1 \) and \( X_2 \) are deformation retracts of \( W \).
THEOREM (3.14) Let $X_1$ and $X_2$ be 3-manifolds in $\mathcal{S}_3'$. If $X_1$ and $X_2$ are h-cobordant then

$\mu(X_1) = \mu(X_2)$.

This follows quickly from the "pasting and smoothing" technique. Indeed, let $X_1 = \partial Y_1$ and $X_2 = \partial Y_2$, where $Y_1$ satisfies (3.1), $i = 1, 2$, and let $X_1$ and $X_2$ co-bound $W$ (i.e., $\partial W = X_1 \cup -X_2$), which gives the cobordant relation. We paste $W$ and $Y_2$ to obtain a manifold, $N$. Notice that we already have the right orientations on the two copies of $X_2$.

Since $X_2$ is a deformation retract of $W$, the manifold $N$ is homotopically equivalent to $Y_2$ and hence $S_N = S_{Y_2}$. On the other hand, $N$ satisfies (3.1) with $\partial N = X_1$. It follows that $\tau(Y_2) = \tau(N) = \tau(Y_1) \pmod{16}$, (cf. (3.3)). This proves (3.14).

It follows that $L(7,1)$ and $L(7,2)$ are not even h-cobordant.
We conclude this section by proving the following lemma, which shows that certain \( \mathbb{Z}_2 \) - homology spheres arise naturally. The higher dimensional analogues are immediate.

**Lemma (3.15)** Let \( Y \) be a simply-connected, torsion-free 4-manifold with boundary, \( \partial Y = X \). If \( \det S_Y \) is odd, \( X \) is a \( \mathbb{Z}_2 \) - homology sphere.

Consider the sequence
\[
H_2(Y) \xrightarrow{J} H_2(Y,X) \xrightarrow{\partial} H_1(X) \to 0
\]
which, as a portion of the homology sequence of the pair \((Y,X)\), is exact (integral homology being understood here). Since \( M \) has no torsion, \( H^2(Y) = \text{Hom}(H_2(Y), \mathbb{Z}) \). We apply the Poincaré-Lefschetz duality,
\[
H_1(Y,X) = H^{4-1}(Y),
\]
to obtain a commutative diagram
\[
\begin{array}{ccc}
H_2(Y) & \xrightarrow{J} & H_2(Y,X) \xrightarrow{\partial} H_1(X) \xrightarrow{i} 0 \\
\downarrow & & \downarrow \\
\text{Hom}(H_2(Y), \mathbb{Z}) & \xrightarrow{\gamma} & H^2(Y)
\end{array}
\]
The homomorphism \( \gamma \) may be interpreted as follows: for each \( a \) in the lattice \( H_2(Y) \) there is an element \( h_a \in \text{Hom}(H_2(Y), \mathbb{Z}) \) given by \( h_a(x) = S_Y(a,x) \), then \( \gamma(a) = h_a \). Obviously \( \gamma \) induces an isomorphism \( \gamma_0 \) from \( \text{coker } \gamma = \text{cokernel of } J \) onto \( H_1(X) \). Since
$H_2(Y)$ and $\text{Hom}(H_2(Y), \mathbb{Z})$ are lattices of the same rank, we have

$$(3.16) \quad H_1(X) \text{ is finite} \iff \det S_Y \neq 0.$$  

Now, $\text{coker } \varrho$ is determined, up to an isomorphism, by the elementary divisors of $\varrho$, we conclude that

$$(3.17) \quad \det S_Y \neq 0 \Rightarrow \text{order of } H_1(X) = |\det S_Y|.$$  

This, in particular, implies (3.15).
1) We introduced the class $\mathcal{Y}_3$ (see p. 3-1) merely for avoiding certain detailed arguments. In fact, $\mathcal{Y}_3$ is precisely the class of all $\mathbb{Z}_2$-homology 3-spheres. To see this we first recall a well-known fact that every 3-dimensional (orientable, differentiable) manifold is parallelizable. Therefore, according to Milnor [2] (see also M.W. Hirsch [1]), every unbounded 3-manifold bounds a (compact) simply connected $n$-manifold. Now let $X$ be any $\mathbb{Z}_2$-homology 3-sphere. We shall show that $X \in \mathcal{Y}_3$. Let $Y$ be a simply connected $n$-manifold such that $\partial Y = X$. Then the condition (3.1a) is trivially fulfilled. Let $M$ be the double of $Y$ (i.e. $M = Y \cup -Y$, obtained by pasting $Y$ and $-Y$ together along their boundaries). It is clear that $S_M$ is unimodular and $w_2(M) = 0$ (since $w_2(Y) = 0$ and $\partial Y = X$ is a $\mathbb{Z}_2$-homology sphere); therefore the quadratic form $S_M$ is even. On the other hand, we have

$$S_M^R(2) = S_Y^R(2) + (-S_Y^R(2))$$

(see p. 3-5). We conclude that $S_Y$ is even and hence $X \in \mathcal{Y}_3$. The following theorem follows easily from our discussion:

**Theorem (3.18)** Let $X$ be a $\mathbb{Z}_2$-homology 3-sphere. If $X$ is embeddable in $\mathbb{R}^4$ then $\mu(X) = 0$. 

The spherical dodecahedral space is a classical example of the Poincaré manifolds (see e.g. [S -T] p.218). The \( \mu \)-invariant of this space, as we shall show later, is \( 1/2 \neq 0 \) and hence it is not embeddable in Euclidean 4-space.

2) The totality of \( \mathbb{Z}_2 \)-homology 3-spheres, \( \mathcal{W}_3 \), is apparently closed under the connected sum operation; i.e., if \( X_1 \) and \( X_2 \) are elements in \( \mathcal{W}_3 \), then the connected sum, \( X_1 \# X_2 \), is again in \( \mathcal{W}_3 \). In fact, \( \mathcal{W}_3 \) endowed with the operation, \( \# \), is a commutative monoid with identity. The invariant, \( \mu \), may be thought of as a map

\[
\mu: \mathcal{W}_3 \rightarrow \mathbb{Q}/\mathbb{Z}.
\]

One proves easily the following

THEOREM (3.19) The map, \( \mu: \mathcal{W}_3 \rightarrow \mathbb{Q}/\mathbb{Z} \), is a monoid-homomorphism (with respect the operation \( \# \) in \( \mathcal{W}_3 \) and the additive operation in \( \mathbb{Q}/\mathbb{Z} \) ) which maps the identity into the identity.

The operation, \( \# \), is compatible with the h-cobordant relation in \( \mathcal{W}_3 \) (see Milnor [2], Lemma 2.3). Therefore \( \mu \) is a homomorphism of the monoid of h-cobordant classes in \( \mathcal{W}_3 \) into \( \mathbb{Q}/\mathbb{Z} \).
4. **Plumbing.**

A fibre bundle, \( \eta \), having total space, \( E \), projection, \( p \), and base space, \( B \), will be denoted by \( \eta = (E, p, B) \). We shall consider mainly bundles with base space \( B = S^2 \). The principal \( SO(2) \)-bundles over \( S^2 \) are classified by \( \pi_1(SO(2)) \cong \mathbb{Z} \), with the Hopf fibration corresponding to a generator in \( \mathbb{Z} \), (see e.g. [S] p. 99 and p. 105; here and in the sequel we shall identify \( SO(2) \) with \( S^1 \)). Therefore, if fixed orientations of \( SO(2) \) and \( S^2 \) are chosen, every principal \( SO(2) \)-bundle over \( S^2 \) corresponds to a unique integer, known as its **characteristic number**.

We recall briefly these notions. Let \( \eta = (E, p, S^2) \) be a given principal \( SO(2) \)-bundle, where \( S^2 \) and \( SO(2) \) are oriented. Choose a base point \( x_0 \in S^2 \) and a neighborhood \( U \) of \( x_0 \), homeomorphic to an open 2-disk. Then the frontier, \( \hat{U} \), of \( U \) is an 1-sphere. The restricted bundles \( E \mid (S^2 \setminus U) \) and \( E \mid \hat{U} \) are both trivial ([S], p. 53). Let \( t : E \mid \hat{U} \rightarrow SO(2) \) be a trivialization and let \( \sigma : (S^2 \setminus U) \rightarrow E \) be a (partial) cross-section. The homotopy class of \( t \sigma \mid \hat{U} : \hat{U} \rightarrow SO(2) \) defines an element in \( \pi_1(SO(2)) = \mathbb{Z} \) (given by the degree of the map \( t \sigma \mid \hat{U} \)) which depends only on the bundle \( \eta \) and the orientations chosen. This integer is then called the **characteristic number** of the bundle.
We now make the orientation convention so that the Hopf fibration has characteristic number $-1$, and we shall denote this bundle by $\xi_{-1} = (X_{-1}, p_{-1}, S^2)$. The Hopf (fibre-) map, $p = p_{-1}$, may be described as follows, ([S], p. 105). Write $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \bar{z}_1 + z_2 \bar{z}_2 = 1\}$ and represent $S^2$ by the complex projective line in which each point is given by an equivalence class $[z_1, z_2]$, $z_1 \cdot z_2 \neq 0$; (the equivalence relation being $(z_1, z_2) \sim \lambda (z_1, z_2)$, $\lambda \neq 0$). We then define $p : S^3 \rightarrow S^2$ by $p(z_1, z_2) = [z_1, z_2]$. For any $[z_1, z_2] \in S^2$, we may find a $(z_1', z_2') \in S^3$ (by normalization) such that $[z_1', z_2'] = [z_1, z_2]$. The fibre $p^{-1}[z_1, z_2]$ is clearly the circle $e^{i\theta}(z_1', z_2')$, $0 \leq \theta \leq 2\pi$. It follows that the action of $\mathbb{Z}_n$ on $S^3$ defined in (3.9), is compatible with the fibration, $p$, if $q = 1$. We recall that in this case (i.e. $q = 1$), the action is given by

$$\gamma(z_1, z_2) = (e^{2\pi i \frac{1}{n}} \cdot z_1, e^{2\pi i \frac{1}{n}} \cdot z_2), \quad \gamma \in \mathbb{Z}_n.$$ 

Passing to the quotient structure, we obtain a bundle $\xi_{-n} = (X_{-n}, p_{-n}, S^2)$, where $X_{-n} = S^3/\mathbb{Z}_n = L(n,1)$. Diagrammatically,

$$\begin{array}{ccc}
S^3 & \xrightarrow{f} & L(n,1) \\
\downarrow & & \downarrow p_{-n} \\
S^2 & & L(n,1)
\end{array}$$
where $f$ is an $n$-fold covering map. We also know that \( \eta_1(L(n,1)) = \mathbb{Z}_n \). Comparing $f$ with the classification theorem of $SO(2)$-bundles cited above, we see that $f$ induces a homomorphism $\eta_1(SO(2)) \xrightarrow{n} \eta_1(SO(2))$, which carries the characteristic number of $\xi_{-n}$ to that of $\xi_{-n}$. This shows that the characteristic number of $\xi_{-n}$ is indeed $-n$, and in this way $\xi_m$ may be constructed for each $m \in \mathbb{Z}$.

Let $D^2$ denote the standard 2-disk and let $SO(2)$ operate on $D^2$ in the obvious fashion. We may consider the 2-disk bundle $\eta_n = (Y_n, p_n, S^2)$ associated with $\xi_n$, (where $p_n : Y_n \to S^2$ is the obvious extension of $p_n : X_n \to S^2$). Then $Y_n$ is a 4-manifold having $X_n = -L(n,1)$ as boundary. The quadratic form of $Y_n$ can be determined as follows. First observe that $H_2(Y_n) = H_2(S^2) = \mathbb{Z}$, where the zero cross-section, $\nu : S^2 \to Y_n$, represents the "positive" generator $g \in H_2(Y_n)$. To determine $S_Y$ it suffices to find $S_Y(g,g) = g \circ g$, the self-intersection number of the class $g$. Take another cross-section $\sigma : S^2 \to Y_n$, also representing $g$. Since $\eta_n \mid (S^2 \setminus x_0)$ is a trivial bundle, we may assume that $\sigma(x) \neq \nu(x)$, if $x \neq x_0$ in $S^2$. Now let $U$ be a neighborhood of $x_0$ as before and let $u : Y_n \xrightarrow{\overline{U}} D^2$ be a trivialization. The degree of the
map $u\sigma \mid (\bar{U} \setminus x_0) : U \setminus x_0 \to D^2 \setminus 0$ is, by definition, the intersection number of $\sigma \mid \bar{U}$ and $\nu \mid \bar{U}$ and, since $\sigma$ and $\nu$ do not intersect elsewhere, this is also the intersection number of $\sigma$ and $\nu$ (which is $g \circ g$).

Identifying $\pi_1(\bar{U})$ with $\pi_1(U \setminus x_0)$ and $\pi_1(SO(2))$ with $\pi_1(D^2 \setminus 0)$, we see that $g \circ g = S_y(g, g)$ is exactly the characteristic number, $n$, of the bundle $\xi_n$.

The manifolds $X_n$ and $Y_n$, $n \in \mathbb{Z}$, are the basic objects for "plumbing" which we shall soon define.

REMARK (4.1) Consider the bundle $\xi_m = (X_m, p_m, S^2)$. Let $W$ be a submanifold of $S^2$ such that the restricted bundle $\xi_m \mid W = (X_w, p_m, S^2)$ is trivial, where $X_w = p_m^{-1}(W)$ and $W$ is oriented in accordance with $S^2$. Let $(x)$ and $(y)$ be given coordinates in $W$ and $S^1$ (the typical fibre) respectively, compatible with the orientations. Then $(x, y)$ may be chosen as coordinates in $X_w$ so that the product structure and orientation of $X_w$ are preserved. Such coordinates in $X_w$ will be called admissible.

A point in $D^2$ may be expressed, in polar coordinates by a pair $(r, x)$, where $0 \leq r \leq 1$ and $x \in \mathbb{R}/\mathbb{Z} = R_1$, the real numbers modulo 1. If $r = 1$, we write $(1, x) = x$; thus a point in $S^1$ is given by $x \in R_1$. Let $U$ be a 2-disk in $S^2$ and let $V = Cl(S^2 \setminus U)$,
which is also a 2-disk. Then both $\mathfrak{T}_m | U$ and $\mathfrak{T}_m | V$ are trivial. Let $(\mathfrak{T}, \xi, \eta)$ and $(\mathfrak{r}, x, y)$ be admissible coordinates in $X_U$ and $X_V$, $(x, y, \xi, \eta \in R_1$). The boundaries $\partial X_U$ and $\partial X_V$ (both homeomorphic to $S^1 \times S^1$) have admissible coordinates $(\xi, \eta)$ and $(x, y)$ respectively. The fact that the characteristic number of $\mathfrak{T}_m$ is $m$ shows that if $f : \partial X_V \to \partial X_U$ is a map defined by $f(x, y) = (\xi, \eta) = (-x, y - mx)$, then the adjunction space $X_V \cup_f X_U$ is precisely $X_m$. The map, $f$, may also be expressed by the matrix

\[
\begin{pmatrix}
-1 & 0 \\
m & 1
\end{pmatrix}.
\]

By a graph we shall mean a finite one-dimensional simplicial complex. A contractible graph will be termed a tree. Thus trees are connected. If two vertices $v_i$ and $v_j$ of a tree are joined by an edge (= 1-simplex we shall denote this (unique) edge by $e_{ij}$. Let $\mathcal{M}$ be a set. A tree, $T$, is said to be weighted by the elements of $\mathcal{M}$ if to each vertex, $v_i$, of $T$ an element, $m_i$ (to be called the weight of $v_i$), of $\mathcal{M}$ is attached. If, in particular, $\mathcal{M} = \mathbb{Z}$, we shall simply say that $T$ is weighted (by integers). A weighted tree will be denoted by $T = (T, m_i)_{1 \leq i \leq s}$; its weighted vertices, by $(v_1, m_1), \ldots, (v_s, m_s)$. The
adjective "weighted" will often be dropped if no confusion is likely to arise.

A plumbing operator, \( P = P_k \), assigns to each tree \( T = (T, m_1) \), weighted by the elements of \( \eta_{2n-1}(SO(2n)) \), a 4k-dimensional manifold \( P(T) \). Here we restrict ourselves to the case \( k = 1 \) and we identify \( \eta_1(SO(2)) \) with \( \mathbb{Z} \). We hope to expound the more general cases later.

1) If the given tree, \( T \), contains a single weighted vertex \( (v, m) \), we define \( P(T) \) to be \( Y_m \), the total space of the bundle \( \eta_m \) defined above.

2) Let \( T \) be a tree consisting of two vertices, \((v_1, m_1), (v_2, m_2)\) and an edge \( e_{12} \). For each \((v_1, m_1)\) we take a copy of \( \eta_{m_1} = (Y_{m_1}, p_{m_1}, S^2), i = 1, 2 \). Choose a 2-disk, \( D_{ij} \), in the base space of \( \eta_{m_1} \) and let \( p_{m_1}^{-1}(D_{ij}) = Y_{ij} \). Since \( \eta_{m_1} \mid D_{ij} \) is trivial, there is a homeomorphism \( u_{ij} : D^2 \times D^2 \rightarrow Y_{ij} \) whose first component gives base coordinates and the second, the fibre coordinates. Let \( t : D^2 \times D^2 \rightarrow D^2 \times D^2 \) be the reflection defined by \( t(x, y) = (y, x) \). Then there is a homeomorphism \( f_{j1} : Y_{j1} \rightarrow Y_{ij} \) given by \( f_{j1} = u_{ij} \cdot tu_{j1}^{-1} \). In our case, we have \( f_{21} : Y_{21} \rightarrow Y_{12} \). Notice that \( Y_{ij} \subset Y_{m_1}(i, j = 1, 2) \) we may paste
\( y_{m_2} \) and \( y_{m_1} \) together along \( y_{21} \) and \( y_{12} \) by means of \( f_{21} \) to obtain a topological 4-manifold \( \tilde{\mathcal{P}}(T) \).

Observe that \( f_{j1} \) fails to be a diffeomorphism only along the "corner" \( u_{j1}(\partial D^2 \times \partial D^2) \), which is a submanifold. According to Milnor [2], we may smooth \( \tilde{\mathcal{P}}(T) \) by straightening the corner to obtain a (differentiable) manifold, \( \mathcal{P}(T) \), and the smoothing is essentially unique. However, the definition of \( \tilde{\mathcal{P}}(T) \) already involves arbitrary choices. It is fairly clear that \( \mathcal{P}(T) \) is independent (up to a diffeomorphism) of the
choice of $u_{ij}$, provided it is admissible. A theorem of R. Thom (Milnor [5]) assures, in particular, that $P(T)$ is independent of the choices of $D_{ij}$.

3) The general pattern of plumbing is now clear. Let $T = (T, m_i)_{1 \leq i \leq s}$ be a tree. For each $(v_i, m_i)$ we take a copy of $\eta_{m_i}$. Suppose that $v_i$ is connected to $v_j, v_k, \ldots$ by edges $e_{1j}, e_{ik}, \ldots$ in $T$. We take, in the base space of $\eta_{m_i}$, 2-disks $D_{ij}, D_{ik}, \ldots$ which are pairwise disjoint and construct homeomorphims $f_{ji}, f_{ki}, \ldots$ as in 2). We then paste $\psi_{m_i}$ and $\psi_{m_j}$, etc., together according to $f_{ji}$, etc., and finally we smooth the resulting topological manifold to obtain $\mathcal{P}(T)$.

While the smoothing process has been described loosely we feel that it may be helpful to give an intuitive picture about straightening the corners (see Milnor [2]). Let $R^+$ denote the positive half-line of $R$. A point, $x$, at the corner, i.e., $x \in u_{1j}(\partial D^2 \times \partial D^2) = u_{1j}(\partial D^2 \times \partial D^2)$, has a neighbourhood which looks like $(\partial D^2 \times \partial D^2) \times (R^+ \times R \cup R \times R^+)$. The second factor may be "straightened" by the transformation

$$(r \cos \theta, r \sin \theta) \mapsto (r \cos \frac{2\theta + \pi}{3}, r \sin \frac{2\theta + \pi}{3})$$

which is differentiable except at $r = 0$. 
Since $Y_m$ has the homotopy type of a 2-sphere, it follows that, for each tree $T = (T,m_i)_{1 \leq i \leq s}$, the 4-manifold $\mathcal{P}(T)$ has the same homotopy type as the one point union of 2-spheres. Thus $\mathcal{P}(T)$ is simply connected and $H_2(\mathcal{P}(T);\mathbb{Z})$ is a lattice of rank $s$.

To a weighted tree $T = (T,m_i)_{1 \leq i \leq s}$ we may associate a quadratic form of rank $s$ whose matrix $M = (\mu_{ij})$ is given in the same fashion as that described in p. 1-9 except $\mu_{11} = m_1$ instead of $\mu_{11} = 2$. Thus the quadratic form given in p. 1-9 is the form associated with $(E_8,m_1)$ where $m_1 = 2$ for $i = 1, \ldots, 8$. It is then clear that the quadratic form of $\mathcal{P}(T)$ is the same as the quadratic form associated with $T$. Observe that the plumbing operator, $\mathcal{P}$, is defined on all (finite) graphs; but in case the graph is not a tree, the resulting manifold will not be simply connected. Plumbing operation may be generalized by allowing other bundles to play the role of $\eta_m$. 
As examples we mention the following trees which arise in the classification theory of simple Lie algebras (see e.g. Séminaire Sophus Lie, 1ère année, exp. 13). These are the only trees, when weighted by 2, whose quadratic forms are positive definite.

\[ A_s \quad v_1 \quad v_2 \quad v_3 \quad \ldots \quad v_s \]

\[ D_s \quad v_1 \quad \quad \quad v_3 \quad \ldots \quad v_s \quad (s \geq 4) \]

\[ E_6 \]

\[ E_7 \]

\[ E_8 \]

In the following table, each vertex of the tree, \( T \), is weighted by 2:

<table>
<thead>
<tr>
<th></th>
<th>( \mu(\partial P(T)) )</th>
<th>( \eta_1(\partial P(T)) = F_T )</th>
<th>( H_1(\partial P(T); \mathbb{Z}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_s</td>
<td>( s/16 ) (s odd)</td>
<td>( C' s+1 )</td>
<td>( \mathbb{Z}_{s+1} )</td>
</tr>
<tr>
<td>D_s</td>
<td></td>
<td>( D''_{s-2} )</td>
<td>( \mathbb{Z}_2 + \mathbb{Z}_2, \text{ s even} ) or ( \mathbb{Z}_4, \text{ s odd} )</td>
</tr>
<tr>
<td>E_6</td>
<td>( 6/16 )</td>
<td>( T' )</td>
<td>( \mathbb{Z}_3 )</td>
</tr>
<tr>
<td>E_7</td>
<td></td>
<td>( W' )</td>
<td>( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>E_8</td>
<td>( 8/16 )</td>
<td>( P' )</td>
<td>0</td>
</tr>
</tbody>
</table>
The groups, $F_T$, are the only finite subgroups of $S^3$ (= unit quaternions). This can be seen as follows. The finite subgroups of $SO(3)$ are known to be $C_n$, $D_n'$, $T$, $W$ and $P$ (the cyclic, the dihedral, the tetrahedral, the octahedral and pentagondodecahedral groups). Regarding $S^3$ as a 2-fold covering group of $SO(3)$, one deduces easily that the only finite subgroups of $S^3$ are the cyclic groups, $C_s'$, and the binary groups $D_8''$, $T'$, $W'$ and $P'$, obtained by lifting the groups $D_s'$, $T$, $W$ and $P$ of $SO(3)$.

The fundamental groups $\pi_1(\mathbb{F}(T))$ and the homology groups $H_1(\mathbb{F}(T))$ can be computed directly by finding out explicitly the generators and relations; however these groups may be obtained much easier from a more general result due to von Randow (v. Randow : Thesis, Bonn, in preparation) which we shall describe briefly. A 3-manifold will be termed a Seifert manifold if it can be fibered, with exceptional fibres, in $S^1$ over $S^2$ (Seifert [1]). To such a manifold, $X$, Seifert associated a system of integers

$$(b; (a_1,b_1), \ldots, (a_r,b_r)),$$

known as Seifert invariants, where $r = \text{number of exceptional fibres}$ and the integers $a_1, b_1$ satisfying the relations $0 < b_1 < a_1$ and $(b_1,a_1) = 1$, $(i = 1, \ldots, r)$. 
Expand \( \frac{a_1}{a_1 - \beta_1} \) into continued fraction:

\[
\frac{a_1}{a_1 - \beta_1} = [\eta_1^{(1)}, \ldots, \eta_s^{(1)}], \quad |\eta_j^{(1)}| \geq 2,
\]

and let \( T \) be the star-shaped tree (weighted by \( \eta_j^{(1)} \)'s):

```
      b+r
     /   \
    /     \
\eta_1^{(1)} -- \eta_1^{(r)} -- \eta_s^{(r)}
     \     
   \eta_s^{(1)}
```

Then \( X = \partial P(T) \). (Von Randow's proof of this fact is based on a special construction of lens spaces via plumbing, which we shall study later in this section.

Notice that lens spaces are special Seifert manifolds.) It is known that for any finite subgroup, \( F \), of \( S^3 \) the coset space, \( S^3/F \), is a Seifert manifold. Indeed, each manifold, \( \partial P(T) \), listed in our table is diffeomorphic to \( S^3/F_T \). For instance, the spherical dodecahedral space, \( S^3/F' \), has Seifert invariants \((-1; (5,1), (3,1), (2,1))\). Here \( r = 3 \) and

\[
b + r = 2
\]

\[
5/(5-1) = 5/4 = [2,2,2,2],
\]

\[
3/(3-1) = 3/2 = [2,2],
\]

\[
2/(2-1) = 2/1 = 2.
\]
From what we discussed above, we get a star-shaped tree
\[ b + r \]
\[ 2 \quad 2 \quad 2 \quad 2 = 2 \quad 2 \quad 2 \]
but this is $E_8$ with each vertex weighted by 2. In other words $\partial \mathcal{P}(E_8)$ is the spherical dodecahedral space if $E_8$ is weighted by 2; therefore $\pi_1(\partial \mathcal{P}(E_8)) = \mathbb{Z}$ and $H_1(\partial \mathcal{P}(E_8)) = 0$. The other lines in the table may be justified in a similar fashion.

Notice that when $T = A_5$ (s odd), $T = E_6$, or $T = E_8$, the manifold $\partial \mathcal{P}(T)$ is a $\mathbb{Z}_2$-homology sphere and consequently the $\mu$-invariant is defined (p. 3-13). The values of $\mu(\mathcal{P}(T))$ are obvious, since the quadratic form associated with each tree $T$ listed above is positive definite and the quadratic form of $\mathcal{P}(T)$ is the same as that of $T$. In particular the $\mu$-invariant of the spherical dodecahedral space is $8/16 = 1/2$ as we claimed toward the end of §3.

As we have seen that in many cases we are interested in the manifold $\partial \mathcal{P}(T)$ rather than $\mathcal{P}(T)$ itself. The assignment, $\partial \mathcal{P} : T \to \partial \mathcal{P}(T)$, will also be called plumbing; it may be defined directly without the help of the operator $\mathcal{P}$. We describe this in three steps:
1') If $T = (v, m)$, we define $\partial \mathcal{P}(T) = \xi_m$.

2') Let $T$ be a tree consisting of $(v_1, m_1)$, $(v_2, m_2)$ and $e_{12}$. Take $D_{12}$ and $D_{21}$ in the base spaces as we did in 2). Let $X_{ij} = P_{m_1}^{-1}(D_{ij})$ and let $u_{ij} : D^2 \times S^1 \to X_{ij}$ be an admissible coordinate function $(i, j = 1, 2)$. Then $\partial X_{ij}$ is equivalent to $S^1 \times S^1$. Define $f_{21} : \partial X_{21} \to \partial X_{12}$ by putting $f = u_1 tu_2^{-1}$, where $t$ is defined as in 2). Let $X'_{ij} = P_{m_1}^{-1}(S^2 \setminus \text{Int } D_{ij})$. Then $\partial X'_{ij} = \partial X_{ij}$ $(i, j = 1, 2)$.

Finally we put

$$\partial \tilde{\mathcal{P}}(T) = X'_{21} \cup f_{21} X'_{12},$$

and then smooth it to obtain $\partial \mathcal{P}(T)$.

3') The general procedure is now clear.

REMARK (4.3) The map, $f_{21}$, may be expressed in terms of the $u_{ij}$-coordinates by a matrix

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
$$

In view of remark (4.1), $\partial \mathcal{P}(T)$ of 2') may be obtained from $X_{12}$ by attaching a copy of $D^2$ to its boundary, $\partial X_{12}$, according to a map $f_{21}$ given by the matrix

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
-1 & 0 \\
-m & 1
\end{pmatrix} = \begin{pmatrix}
-m & 1 \\
-1 & 0
\end{pmatrix}.$$
Consider the weighted tree \((A_s, m_i)_1 \leq i \leq s\):

\[ m_1 \, m_2 \, m_3 \, \ldots \, m_{s-1} \, m_s, \]

where \(m_i\)'s are integers. We propose to study \(\partial \mathcal{O}(A_s)\). Take a copy of \(\xi_m \) for each \((v_i, m_i)\) in \(A_s\) and let the base space of \(\xi_m\) be denoted by \(S^2_i\) \((i = 1, \ldots, s)\). In each \(S^2_i\) (represented by extended complex numbers) take \(D_{i,i+1}\) and \(D_{i,i-1}\) centered at 0 and \(\infty\) respectively. Let \(\Delta_1 = S^2_1 \setminus D_{1,i+1} \setminus D_{1,i-1}\) \((i = 2, \ldots, s-1)\), \(\Delta_1 = S^2_1 \setminus D_{12}\) and \(\Delta_s = S^2_s \setminus D_{s-1,s}\). We may consider \(\Delta_1\) as obtained from the space \(\Delta'_1 = D^2 \setminus D_{12}\) by collapsing \(\partial D^2\) to a point \(\infty\). Let

\[ L = \infty \times S^1 \cup f_1 \Delta_1' \times S^1 \cup f_2 \Delta_2 \times S^1 \cup f_3 \ldots \cup f_s \Delta_s \times S^1, \]

where \(f_1 : \partial D_{i,i-1} \times S^1 \to \partial D_{i-1,i} \times S^1\) \((f_1 : \partial D_{21} \times S^1 \to S^1)\)

Parametrize \(\partial D_{ij} \times S^1 = S^1 \times S^1\) by a pair of numbers \((x, y)\) in \(\mathbb{R}/\mathbb{Z}\). From remarks (4.1) and (4.3) it is
easily seen that $L$ is homeomorphic to $\partial \mathcal{P}(A_s)$ if the maps, $f_j$, are given by the matrices:

\[
f_j : \begin{pmatrix} -m_j & 1 \\ -1 & 0 \end{pmatrix} \quad (j = 2, \ldots, s)
\]

\[
f_1 : \begin{pmatrix} -m_1 & 1 \\ 0 & 0 \end{pmatrix}
\]

Furthermore, in this case, $L$ (after smoothing) is diffeomorphic to $\partial \mathcal{P}(A_s)$. It is also easy to see that $L \cong S^1 \cup_f D^2 \times S^1$, where $f : \partial D^2 \times S^1 \to S^1$ is described by a matrix

\[
(4.6) \begin{pmatrix} n & q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -m_1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -m_2 & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -m_s & 1 \\ -1 & 0 \end{pmatrix}.
\]

Observe that

\[
(4.6a) \begin{pmatrix} -m_1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -m_2 & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -m_s & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} n & q \\ -q' & p \end{pmatrix},
\]

where the integers $n$ and $q$ are the same as those in $(4.6)$. Notice also that each factor in the left-hand side has determinant $= +1$, hence the determinant of the matrix $\begin{pmatrix} n & q \\ -q' & p \end{pmatrix}$ in the right-hand side is $+1$.

This shows

\[
(4.7) \quad np + qq' = 1,
\]

and it follows that $n$ and $q$ are co-primes.

Recall that a well-known description of the lens space $L(n,q)$ is given by
\[ L(n, q) = S^1 \cup_{g} D^2 \times S^1, \]

where \( g : \emptyset D^2 \times S^1 = S^1 \times S^1 \rightarrow S^1 \) maps a point \((x, y)\) into \( nx + qy \) \((x, y) \in \mathbb{R}/\mathbb{Z}\). In other words, the matrix of \( g \) is exactly

\[
\begin{pmatrix}
  n & q \\
  0 & 0
\end{pmatrix}
\]

We have

**THEOREM (4.8)** Let \( A_s = (A_{s, m_1})_{1 \leq 1 \leq s} \). Then

\[ \delta^p(A_g) = L(n, q), \]

where \( n \) and \( q \) are defined as in (4.6).

If we reverse the order of the tree \((A_{s, m_1})\) and plumb according to the reversed tree we get \( L(n, q') \), where \( q' \) is given in (A.6a). From (4.7) it follows that \( qq' = 1 \pmod{n} \). Our result is therefore compatible with the classical theorem concerning the lens spaces (see e.g. [S-T] p. 215 Satz II).

For given co-primes \( n \) and \( q \), \( 0 \leq q \leq n \), we can find a tree \((A_{s, m_1})_{1 \leq 1 \leq s}\) such that \( \delta^p(A_g) = L(n, q) \). To see this we let \( \lambda_0 = n, \lambda_1 = q \), and use Euclidean algorithm to get

\[
\begin{align*}
\lambda_0 &= a_1 \lambda_1 - \lambda_2, & 0 \leq \lambda_2 < \lambda_1, & a_1 > 1, \\
\lambda_1 &= a_2 \lambda_2 - \lambda_3, & 0 \leq \lambda_3 < \lambda_2, & a_2 > 1, \\
&\vdots & & \vdots \\
\lambda_{s-1} &= a_s \lambda_s - \lambda_{s+1}, & \lambda_{s+1} = 0, & \lambda_s = 1, a_s > 1.
\end{align*}
\]
This is equivalent to saying that \( n/q \) can be expanded into a continued fraction \([a_1,a_2,\ldots,a_s]\), i.e.

\[
n/q = a_1 - \frac{1}{a_2 - \frac{1}{\ddots - \frac{1}{a_s}}}\]

It is clear from (4.9) that

\[
\begin{pmatrix}
\lambda_s & \lambda_{s+1} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
a_s & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
a_{s-1} & 1 \\
-1 & 0
\end{pmatrix}
\cdots
\begin{pmatrix}
a_1 & 1 \\
-1 & 0
\end{pmatrix}
= \begin{pmatrix} n \\ q \end{pmatrix}.
\]

Since

\[
\begin{pmatrix}
\lambda_s & \lambda_{s+1} \\
0 & 1
\end{pmatrix}
= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

is the identity matrix, we deduce that

\[
\begin{pmatrix}
a_s & 1 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
a_{s-1} & 1 \\
-1 & 0
\end{pmatrix}
\cdots
\begin{pmatrix}
a_1 & 1 \\
-1 & 0
\end{pmatrix}
= \begin{pmatrix} n \\ q \end{pmatrix}.
\]

Comparing this with (4.6) we conclude that if

\( A_s = (A_s,m_i)_{1 \leq i \leq s} \) is a tree with weights \( m_i = -a_{s-i+1} \) \((1 \leq i \leq s)\), then \( \partial \mathcal{P}(A_s) = L(n,q) \).

The lens space, \( L(n,q) \) obtained in the preceding construction bounds a 4-manifold, \( \mathcal{P}(A_s) \), whose quadratic form is the same as that of \( (A_s,m_i) \) with \( m_i = -a_{s-i+1} \). As the integers \( a_i \) may not be even, this quadratic form does not lend itself to the computation of \( \mu(L(n,q)) \) in case the latter is defined (cf. p. 3-1). This situation can be remedied. According to \( \S 3 \), the \( \mu \)-invariant, \( \mu(L(n,q)) \), is only defined for odd \( n \) and in this case we may,
without loss of generality, assume that \( q \) is even.

For such \( n \) and \( q \), (4.9) may be modified to yield:

\[
\begin{align*}
\lambda_0 &= n, \quad \lambda_1 = q \\
\lambda_0 &= b_1 \lambda_1 - \lambda_2, \quad |\lambda_2| < |\lambda_1| \quad |b_1| > 0, \\
&\vdots \quad \vdots \\
\lambda_{s-1} &= b_s \lambda_s - \lambda_{s+1}, \quad \lambda_{s+1} = 0, |\lambda_s| = 1, \quad |b_s| > 0,
\end{align*}
\]

(4.9a)

where each \( b_1 \) is even. This, by the way, proves an assertion announced in p. 3-7. Let now \( A_s = (A_s, n_i)_{1 \leq i \leq s} \) have weights \( n_i = -b_{s-i+1} \). It is easy to show that \( \mathcal{P}(A_s) = L(n,q) \) in spite of the fact that \( \lambda_s \) may take the value -1. In this construction, \( L(n,q) \) bounds a 4-manifold, \( Y = \mathcal{P}(A_s) \), whose quadratic form, \( S_Y \), is even. This proves (3.10).

With respect to an obvious base \( (e_i)_{1 \leq i \leq s} \) in \( H_2(Y;\mathbb{Z}) \), \( S_Y \) is represented by the matrix

\[
\begin{pmatrix}
-b_1 & 1 & & 0 \\
& 1 & -b_2 & \\
& & \ddots & \ddots & 1 \\
0 & & & 1 & -b_s
\end{pmatrix}.
\]

Let \( M \) denote the matrix

\[
(4.10) \quad M = \begin{pmatrix}
b_1 & 1 & & 0 \\
& 1 & b_2 & \\
& & \ddots & \ddots & 1 \\
0 & & & 1 & b_s
\end{pmatrix}.
\]
Then, relative to the base \((-1)^{l+1} e_i\)_{1 \leq i \leq s} in \(H_2(Y; \mathbb{Z})\), \(S_y\) is given by \(-M\). To prove the recipe (3.12), it remains to show that \(\tau(M) = p^+ - p^-\), where \(p^+\) (resp. \(p^-\)) is the number of positive (resp. negative) diagonal entries of \(M\). In fact, we shall prove the following lemma.

**LEMMA (4.11)** Let \(M\) be a matrix of the form described in (4.10) with \(|b_i| > 1\). Then \(\tau(M) = p^+ - p^-\).

It can be shown by an easy induction that \(M\) is congruent over \(\mathbb{R}\) to a diagonal matrix diag \((c_1, \ldots, c_s)\), where

\[
(4.12) \quad c_1 = b_1 - \frac{1}{b_1 + 1 - \cdots - \frac{1}{b_s}} = [b_{i+1}, \ldots, b_s].
\]

If \(|b_i| > 1\), then \(\text{sgn} c_1 = \text{sgn} b_1\). This obviously proves (4.11), and hence (3.12).

We end this section by a digression. Let \(n\) be an odd integer and let \(q\) be an integer prime to \(n\). Suppose that \(n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_t^{\beta_t}\) is the canonical decomposition of \(n\) (into prime factors). The Jacobi symbol, \((q|n)\), may be defined through the Legendre symbol, \((q|p_i)\), by the formula

\[(q|n) = (q|p_1)^{\beta_1} \cdots (q|p_t)^{\beta_t} .\]
It is clear that, if \( q \) and \( q' \) are integers such that \( qq' \) is a quadratic residue modulo \( n \) (i.e. \( qq' = x^2 \) has a solution in \( \mathbb{Z}_n \)), then
\[
(q|n) = (q'|n).
\]
From Theorem (3.13), we see that \( (q|n) \) is an orientation-preserving homotopy type invariant on the category of lens spaces \( L(n,q) \) with odd \( n \). Notice that the h-cobordant invariant \( \mu L(n,q) \) is also defined for such spaces. We have
\[
(q|n) = (-1)^{4\mu L(n,q)} + (n-1)/4;
\]
that is the Jacobi symbol, \( (q|n) \), for a fixed \( n \) is completely determined by \( \mu L(n,q) \).

To prove (4.14) we first observe that, by using the identities
\[
(-1|n) = (-1)^{(n-1)/2}
\]
and \( \mu L(n,q) = -\mu L(n,n-q) \), we can restrict ourselves to the case where \( q \) is even. Expand \( n/q \) into continued fraction: \( n = [b_1, b_2, \ldots, b_s] \) according to (4.9a), i.e., \( |b_1| > 1 \) and \( b_i \) even for \( i = 1, 2, \ldots, s \). We have shown that \( L(n,q) \) bounds a 4-manifold \( \mathbb{P}(\mathcal{A}_g) = \mathcal{Y} \) whose quadratic form, \( S_\mathcal{Y} \), may be described as follows. Let \( S = -S_\mathcal{Y} \); then the matrix of \( S \) is given by (4.10). In other words,
\[(4.16) \quad S = \sum_{i=1}^{s} b_i x_i^2 + 2 \sum_{i=1}^{s} x_i x_{i+1} \sim \sum_{i=1}^{s} c_i y_i^2\]

where the rational numbers, \(c_i\), are given by \((4.12)\).

Therefore \(\mu L(n,q) = -\tau(S_1)/16 = \tau(S)/16 = (p^+ - p^-)/16\)
(this number being understood to be reduced modulo 1).

We now compute the Hasse-Minkowski symbol \(c_p(S)\) for all odd primes \(p\).

**Case I.** \(p \nmid n\).

From \((3.17)\) we see that \(\det S = \pm n\). Our assumption then implies that \(p \nmid \det S\) and hence \(c_p(S) = 1\), by \((0.24)\).

**Case II.** \(p | n\).

In this case \(p\) is a prime factor of \(n\), say \(p = p_1\) and \(n = p_1^{\beta_1} \cdots p_t^{\beta_t}\). Since \((q,n) = 1\), we get \(p \nmid q\). Write \((4.16)\) in the form

\[S = \frac{n}{q} y_1^2 + \sum_{i=2}^{s} c_i y_i^2 = S_1 + S_2,\]

where \(S_1 = \frac{n}{q} y_1^2 = c_1 y_1^2\) and \(S_2 = \sum_{i=2}^{s} c_i y_i^2\).

It follows that \(\det S_2 = \pm q\) and, since \(p \nmid q\),
\(c_p(S_2) = 1\).

Now \(S_1\) is an unary form \(c_p(S_1) = 1\). Therefore
\[c_p(S) = c_p(S_1) c_p(S_2)(\det S_1, \det S_2)_p = (\frac{n}{q}, q \cdot \text{sgn } \det S_2)_p.\]
Since $p = p_1$ and $(n,q) = 1$, we may write $n/q = p^{\beta_1} \alpha_1$ and apply the Property 5) of Hilbert symbol (see p. 0-14) to obtain

$$(\frac{n}{q}, q, \text{sgn det } S_2) = (q \mid \text{sgn det } S_2 \mid p)^{\beta_1}$$

$$= (q \mid p)^{\beta_1} (\text{sgn det } S_2 \mid p)^{\beta_1}.$$ 

This, together with the conclusion from Case I, shows that

$$\prod_{\text{odd prime}} c_p(S) = \prod_{i=1}^{t} (q \mid p_1)^{\beta_1} (\text{sgn det } S_2 \mid p_1)^{\beta_1}$$

$$= (q \mid n)(\text{sgn det } S_2 \mid n).$$

We may assume that $n/q > 0$; therefore $\text{sgn det } S_2 = \text{sgn det } S$. Applying Lemma (0.25) we get

$$c_2(S)c_\infty(S) = (q \mid n)(\text{sgn det } S \mid n).$$

Since $S$ is even, Theorem (1.9) gives

$$c_2(S)c_\infty(S) = (-1)^{(\sum + \text{det } S - \text{sgn det } S)/4}.$$ 

Combining (4.15), (4.17) and (4.18) we obtain (4.14).

The lecturer is indebted to T.W.S. Cassels for a very helpful letter containing essentially the preceding proof. See also a forthcoming paper of Cassels in the Comm. Math. Helv..
5. "Riemann Surfaces" of Complex Dimension 2.

Let $M = M^{(n)}$ be a complex manifold of complex dimension $n$, not necessarily compact. (Complex dimensionality will be expressed by a super-script in parentheses.) A complex analytic subset of $M$ is said to be of maximal dimension if it has complex co-dimension 1. We recall that such a subset, $N$, is given locally by the set of zeros of a holomorphic function. More precisely, if $p$ is a point in $N$ then there is an open neighborhood, $U$, of $p$ in $M$ and a holomorphic function, $f$, defined on $U$ (not identically zero on any connected-component of $U$) such that a point $q$ of $U$ belongs to $N \cap U$ if, and only if, $f(q) = 0$. The point $p \in N$ is termed regular if the above mentioned function, $f$, may be chosen as a coordinate function about $p$; otherwise $p$ is singular.

Let $\{f_i\}_{i \in I}$ be an indexed family of meromorphic function in which each function, $f_i$, is defined on an open subset, $U_i$, of $M$, $(i \in I)$ such that

a) $\{U_i\}_{i \in I}$ forms an open covering of $M$,

b) $f_i$ is not identically zero on any connected-component of $U_i$, and

c) $f_i/f_j$ is holomorphic and $\neq 0$ in $U_i \cap U_j$. 

As we know, such a family, \( \{f_i\} \), of functions defines a divisor, \( D \), of \( M \). We shall express this situation in symbol by \( D \sim \{f_i\}_{i \in I} \). In virtue of the condition c), the locus of zeros and poles of a divisor is well-defined. This locus, together with multiplicities of its irreducible components, will often be identified with the divisor itself. If, in particular, each function in the family, \( \{f_i\} \), is holomorphic, the corresponding divisor, \( D \sim \{f_i\} \), is termed **non-negative**. A non-negative divisor, \( D \), is called **positive** if its locus of zeros, \( N_D \), is non-empty; in such case, \( N_D \) is a complex analytic subset of maximal dimension. Perhaps the simplest non-negative divisor is one given by a single, globally defined holomorphic function, i.e., \( D \sim \{f\} \). For such a divisor, we shall simply write \( D = \{f\} \) and \( N_D = \{ f = 0 \} \).

To each divisor, \( D \sim \{f_i\}_{i \in I} \), there is an associated holomorphic complex line bundle, \( [D] \), of \( M \) given by the transition functions

\[ f_{i,j} = f_i/f_j \quad \text{in} \quad U_i \cap U_j. \]
The characteristic class, \( c_1(D) \), of \( D \) is then defined to be the Chern class, \( c_1([D]) \), of the bundle \([D]\).

If \( D = (f) \), then \([D]\) is trivial and hence \( c_1(D) = 0 \).

Let \( D \) be a positive divisor of \( M \) and let \( N_D \) be its locus of zeros. Each irreducible component of \( N_D \) has a fundamental homology class in the sense of Borel-Haefliger [1]. Therefore \( D \), considered as \( N_D \) together with its multiplicities, represents a homology class, \( h(D) \), in \( \mathcal{H}_{2n-2}^\rho(M;\mathbb{Z}) \), where the script \( H \) denotes homology with closed supports. We shall reserve \( H \) to denote the ordinary homology. Borel and Haefliger proved that

**Lemma (5.1)** If \( D \) is a positive divisor then the homology class \( h(D) \) in \( \mathcal{H}_{2n-2}^\rho(M;\mathbb{Z}) \) is dual to the characteristic class \( c_1(D) \) in \( H^2(M;\mathbb{Z}) \).

Let \( V \) and \( W \) be \( \mathbb{Z} \)-modules. A bilinear form

\[ g: V \times W \rightarrow \mathbb{Z} \]

may be defined by means of a correlation \( \mathcal{S}: V \rightarrow W^* \) (cf. p. 0-1), where \( W^* \) is the \( \mathbb{Z} \)-dual of \( W \). Let us denote by \( H^1, \overline{H}_1 \) and \( \overline{H}^1 \) the appropriate homology and cohomology groups modulo torsion, e.g. \( \overline{H}_1^\rho(M;\mathbb{Z}) = \mathcal{H}_1^\rho(M;\mathbb{Z})/(\text{torsion}) \). For a compact manifold, \( M = M^m \), the \( \mathbb{Z} \)-modules \( \overline{H}_{m-1}(M;\mathbb{Z}) = V \), \( \overline{H}_1(M;\mathbb{Z}) = W \) are lattices and \( W^* = \overline{H}^1(M;\mathbb{Z}) \). The Poincaré duality map
$H_{m-1}(M;\mathbb{Z}) \to H^1(M;\mathbb{R})$ induces a homomorphism \( \mathcal{J} : \mathcal{V} \to \mathcal{W}^* \)
which, taken as a correlation, defines a bilinear form
\[
S : \overline{H}_{m-1}(M;\mathbb{Z}) \times \overline{H}_1(M;\mathbb{Z}) \to \mathbb{Z}.
\]
The integer \( S(x,y) \) is precisely the intersection number of the homology classes \( x \) and \( y \). This notion may be generalized to non-compact manifolds. In fact, if \( M^n \) is non-compact, then the \( \mathbb{Z} \)-modules \( \overline{H}_{m-1}(M;\mathbb{Z}) = \mathcal{V} \)
and \( \overline{H}_1(M;\mathbb{Z}) \) are torsion-free and \( \overline{H}^1(M;\mathbb{Z}) \subseteq \mathcal{W}^* \).
The Poincaré duality map \( H_{m-1}(M;\mathbb{Z}) \to H^1(M;\mathbb{Z}) \) again induces a correlation \( \mathcal{J} : \mathcal{V} \to \mathcal{W}^* \) and hence defines a bilinear form
\[
S : \overline{H}_{m-1}(M;\mathbb{Z}) \times \overline{H}_1(M;\mathbb{Z}) \to \mathbb{Z}.
\]
In this fashion the intersection number of a class \( x \in \overline{H}_{m-1}(M;\mathbb{Z}) \) and a class \( y \in \overline{H}_1(M;\mathbb{Z}) \) is defined, i.e. \( x \circ y = S(x,y) = y \circ x \). The intersection of cycles can then be defined by means of the homology classes they represent.

Let \( D = (f) \) be a divisor of \( M^{(n)} \). According to \((5.1)\), \( \mathcal{J}(h(D)) = c_1(D) = 0 \); this proves

**COROLLARY (5.2)** If \( D = (f) \), then the intersection number \( h(D) \circ x \) vanishes for each \( x \in H_2(M;\mathbb{Z}) \).

(Notice that if \( x \) is a torsion class, \( h(D) \circ x \) is by definition zero.) If we consider \( D \) as a cycle in \( M \),
then for any cycle, \( y \), with compact support in \( M \),
\[ D \cdot y = 0. \]

Now, let \( p \) be a point in the complex manifold \( M = M^{(n)} \). The complex line elements at \( p \) in \( M \) form a complex projective space \( \mathbb{CP}^{(n-1)} \). "Replacing" the point \( p \) of \( M \) by a copy of \( \mathbb{CP}^{(n-1)} \) we obtain a new complex manifold \( \sigma_p M \); i.e., \( \sigma_p M = (M \setminus p) \cup \mathbb{CP}^{(n-1)} \).

The assignment, \( M \rightarrow \sigma_p M \), is known as the \( \sigma \)-process, blowing-up operation or a quadratic transformation. To make the term "replacing" precise, we take a coordinate neighborhood, \( U \), of \( p \) in \( M \) with local coordinates \( z_1, \ldots, z_n \) centered at \( p \) (i.e. \( p = (0) \)). Let \( \mathbb{CP}^{(n-1)} \) be represented by homogeneous coordinates \( w_1, \ldots, w_n \).

There is an obvious map, \( \mathcal{F} \), from \( U \setminus (0) \) onto \( \mathbb{CP}^{(n-1)} \). The graph, \( \Gamma \), of \( \mathcal{F} \) in \( U \times \mathbb{CP}^{(n-1)} \) together with \( K_p = (0) \times \mathbb{CP}^{(n-1)} \) forms a non-singular analytic subset, \( N \), of \( U \times \mathbb{CP}^{(n-1)} \). Indeed, if \( q = (w_1, \ldots, w_n) \in \mathbb{CP}^{(n-1)} \) is such that \( w_1 \neq 0 \), then \( (0) \times q \) has a coordinate neighborhood in \( U \times \mathbb{CP}^{(n-1)} \) with local coordinates \( z_1, \ldots, z_n, w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n \). With respect to these coordinates a point \( r = (z_1, \ldots, z_n, \ w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n) \) belongs to \( N \) if, and only if,
\[ z_j = z_1 w_j \quad (1 \leq j \leq n, \ j \neq 1). \]

Thus \( w_1, \ldots, w_{i-1}, z_1, w_{i+1}, \ldots, w_n \) constitute a system of
local coordinates in $N$ about $(0) \times q$. We then identify $U \setminus (0)$ with $\Gamma = N \setminus K_p$ to obtain $\sigma_p M$.

It can be shown that $\sigma_p M$ does not depend on the choice of the neighborhood, $U$, used to define it and depends only on $M$ and the point $p$.

There is a projection $\pi_p : \sigma_p M \to M$ which maps $K_p$ into $p$ and maps $\sigma_p M \setminus K_p$ homeomorphically onto $M \setminus p$.

We now restrict ourselves to the case where $\dim c = 2$. Let $p \in M$, having local coordinates $z_1, z_2$ (centered at $p$) defined on a neighborhood, $U$, of $p$ in $M$. In the complex manifold $\sigma_p M$, these coordinates are replaced by two systems of local coordinates, $u, v$ and $\tilde{u}, \tilde{v}$, such that

\begin{align*}
\begin{cases}
    z_1 = u \\
    z_2 = uv
\end{cases} & \quad (z_1 \neq 0), \\
\begin{cases}
    z_1 = \tilde{u}\tilde{v} \\
    z_2 = \tilde{v}
\end{cases} & \quad (z_2 \neq 0),
\end{align*}

as one can see from (5.3). The subspace $K_p$ of $\sigma_p M$ is obviously given by the equations $u = 0$ and $\tilde{v} = 0$.

It is also clear that topologically $K_p$ is a 2-sphere embedded in $\sigma_p (M)$ and hence it represents a cycle with compact support in $\sigma_p (M)$. 
LEMMA (5.5) The self-intersection number $K_p \circ K_p$ is $-1$.

Indeed, without changing of notations, $z_1$ may be considered as a function defined on $\mathcal{C}_p^{-1}(U)$. (Strictly speaking, $z_1$ is a function defined on $U$ only. When we consider $z_1$ as a function on $\mathcal{C}_p^{-1}(U)$ we actually mean the function which is expressed in $(u,v)$-chart by $u$ and in $(\tilde{u},\tilde{v})$-chart by $\tilde{u}\tilde{v}$ (see (5.4)). Similar remarks apply to (5.7).) The divisor $|z_1 = 0|$ in $\mathcal{C}_p^{-1}(U)$ consists of $K_p$ and $|\tilde{u} = 0|$; in other words $|z_1 = 0| = K_p + |\tilde{u} = 0|$. Since the intersection number $K_p \circ |\tilde{u} = 0|$ is obviously $+1$. Applying (5.2) to the open manifold $\mathcal{C}_p^{-1}(U)$ we get

$$0 = K_p \circ |z_1 = 0| = K_p \circ K_p + K_p \circ |\tilde{u} = 0|.$$ 

This clearly proves the lemma.

REMARK (5.6) If we ignore the complex analytic structure on $M$ and consider $M = M^4$ as a $C^\infty$-manifold, then $\sigma_p(M)$ is $C^\infty$-equivalent to the connected sum $M \# (-C_p(2))$.

The blowing-up operation may be iterated. Let $p_1$ be a point in $M$ and consider $\mathcal{C}_{p_1} M$. Pick a point $p_2$ in $K_{p_1} = \mathcal{C}_p^{-1}(p)$. We may construct a complex manifold $\sigma_{p_2} \circ_{p_1} M = \sigma_{p_1 p_2} M$. Denote $K_{p_1}$ by $K_1$, then $K_1$ and $K_2$ intersect at exactly one point. There is a
projection, \( \tau_{p_1p_2} \circ \sigma_{p_1p_2} : M \to M \), which maps \( K_1 \cup K_2 \) into \( p \). Take a point \( p_3 \) in \( \tau_{p_1p_2}^{-1} = K_1 \cup K_2 \) and construct \( \sigma_{p_3} (\sigma_{p_1p_2} M) = \sigma_{p_1p_2p_3} M \), etc. Finally we arrive at a complex manifold \( \tilde{M} = \sigma_{p_1 \ldots p_s} M \) and \( \tilde{\tau} = \tilde{M} \to M \) with \( \tilde{\tau}^{-1}(p_1) = K_1 \cup \ldots \cup K_s \). Notice that \( K_i \) and \( K_j \) (\( i \neq j \)) are either disjoint or intersecting at one point regularly.

Thus the "spherical space", \( \tilde{\tau}^{-1}(p_1) \), is dual to a tree, \( T \), in the sense that each \( K_i \) corresponds to a vertex, \( v_i \), of \( T \) and \( K_i \) intersects \( K_j \) (\( i \neq j \)) if, and only if, \( v_i \) and \( v_j \) are joint in \( T \) by an edge \( e_{ij} \).

We may weight the tree \( T \) so that \( v_i \) is weighted by the self-intersection number of \( K_i \) in \( \tilde{M} \). The weighted tree so obtained is called the dual weighted tree of the spherical space \( \tilde{\tau}^{-1}(p_1) \). The self-intersection number of \( K_i \) can easily be computed by applying (5.2) to the open manifold \( \tilde{\tau}^{-1}(U) \), where \( U \) is a coordinate neighborhood of \( p_1 \).

We are now ready to investigate how the resolution of singularities in the "Riemann surface" of an algebraic function by blowing-up process is related to the plumbing operation studied in \( \S 4 \). We believe that an example will be sufficient to clarify the situation.

**EXAMPLE (5.7)** Let \( M = \mathbb{C}^2 \) and let \( f = \sqrt[3]{z_1} + z_2^4 \) be defined on \( M \). The origin \( p_1 = (0,0) \) is the only
non-uniformizable singularity of $f$. Let $w = z_1^3 + z_2^4$; then the locus $|w = 0|$ is a complex analytic subset along which the branching of $f$ occurs. Blow-up $M$ at the point $p_1$ and in $\sigma_{p_1}M$ consider the local coordinates $u, v$ and $\tilde{u}, \tilde{v}$ given by (5.4). $w$ may be considered as a function defined on $\sigma_{p_1}M$ (see the proof of (5.5)). The divisor $|w = 0|$ in $\sigma_{p_1}M$ is expressed by $|\tilde{v}^3(\tilde{v} + \tilde{u}^3) = 0|$ and $|u^3(1 + u^4v) = 0|$. The last two loci coincide except at one point. In fact, $|\tilde{v} = 0|$ and $|u = 0|$ give the 2-sphere $K_1 = K_{p_1}$ in $\sigma_{p_1}M$ and $|(1 + u^4v) = 0| = |(\tilde{v} + \tilde{u}^3) = 0|$. We represent this divisor by a diagram:

$$\begin{array}{c}
\tilde{v} + \tilde{u}^3 = 0 \\
\downarrow \\
3 \\
K_1
\end{array}$$

where the integers, 1 and 3, expresses the multiplicity of the irreducible components indicated. Now the non-uniformizable singularity of $|w = f$ (consider as function on $\sigma_{p_1}M$) is at the point $p_2$ where $(\tilde{u}, \tilde{v}) = 0$. Since this point does not appear in the $(u,v)$-chart, we may consider the restriction of $w$ in $(\tilde{u}, \tilde{v})$-chart i.e. we consider only $\tilde{v}^3(\tilde{v} + \tilde{u}^3)$. Blow-up $\sigma_{p_1}M$ at the point $p_2$ and in $\sigma_{p_1p_2}M$ we take local coordinates $u_1, v_1$.
satisfying \( \tilde{u} = u_1, \tilde{v} = u_1 v_1 \) (for reasons similar to that given above, the other chart may be discarded). In this chart \( |w = 0| \) is given by \( |u_1^4 v_1^3 (v_1 + u_1^2) = 0| \) or in diagram:

Next, let \( p_3 \) be the point where \( (u_1, v_1) = (0, 0) \).

Blow-up at \( p_3 \), \( (u_1 = u_2, v_1 = u_2 v_2; u_1 = \tilde{u}_2 \tilde{v}_2, v_1 = \tilde{v}_2) \), we see \( |w = 0| \) is given by \( |u_2^8 v_2^3 (v_2 + u_2) = 0| \) and \( |\tilde{u}_2^4 \tilde{v}_2^8 (1 + \tilde{u}_2^2 \tilde{v}_2) = 0| \).

Finally we blow-up at \( p_4 \) where \( (u_2, v_2) = (0, 0) \) to obtain the manifold \( \tilde{M} = \sigma_{p_1 p_2 p_3 p_4} \) in which the divisor \( |w = 0| \) is expressed by \( |u_3^{12} v_3^3 (v_3 + 1) = 0| \) and \( |\tilde{u}_3 \tilde{v}_3^{12} (1 + \tilde{u}_3) = 0| \). In diagram we have:
Consider the function $f = \sqrt{w}$ in $\tilde{M}$; it is now uniformizable. Branching of $f$ only occurs along $K_1$ and $L = |v_3 + 1 = 0|$ where the multiplicities are odd. In the Riemann surface (of complex dimension 2) of $f = \sqrt{w}$, $K_4$ is lifted into a 2-sphere, $K'_4$, with two branches, $K'_1$ and $L'$ sticking out ($K'_4$ covers $K_4$ twice, $K'_1$ covers $K_1$ once and $L'$ covers $L$ once). Thus the divisor $D = |f = 0|$ of the Riemann surface can be expressed by the diagram

where $K'_3$ and $K'_5$ cover $K_3$; $K'_2$ and $K'_6$ cover $K_2$. All loci except $L'$ are 2-spheres. The dual diagram of the union of 2-spheres, $\bigcup K'_i$, is precisely $E_6$ (with vertices re-named):
We now compute the self-intersection numbers, $K_i \cdot K_i$ (1 = 1, ..., 6), by means of (5.2). First notice that if $K_i$ and $K_j$ are not disjoint then they intersect at one point regularly and hence $K_i \cdot K_j = +1$. We have, for example,

$$0 = K_i \cdot D = 6 + 3K_i \cdot K_i,$$

$$0 = K_j \cdot D = 3 + 1 + 4 + 4 + 6K_j \cdot K_j.$$

Therefore $K_i \cdot K_i = K_j \cdot K_j = -2$. The reader will have no trouble to show that $K_i \cdot K_i = -2$ for each $i = 1, 2, ..., 6$. The dual weighted tree of the union of 2-spheres, $K = \cup K_i$, is $E_6$ weighted by -2. Let us denote this tree by $(E_6;-2)$; $E_6$ weighted by +2 is then denoted by $(E_6;2)$. It is now clear that if $V$ is a tubular neighborhood of $K$ in the Riemann surface of (the modified) $f$, then $V$ is diffeomorphic to $\mathcal{P}(E_6;-2)$. The last space is by no means new; indeed, up to orientation it is the same as the space $\mathcal{P}(E_6;2)$ which we studied in $\mathcal{S}^4$. In particular, $\partial V \approx -(S^3/T')$. 
We now return to the function \( f = \sqrt{z_1^3 + z_2^4} \) on \( \mathbb{C}^2 \). Let \( \psi \) be the projection from the Riemann surface, \( R_f \), of \( f \) onto \( \mathbb{C}^2 \). Then \( \psi^{-1}(0) = q \) is a singular point of \( R_f \). Let \( B \) be the unit ball in \( \mathbb{C}^2 \), then \( \psi^{-1}(B) \) is a neighborhood of \( q \). The boundary of this neighborhood is obviously diffeomorphic to \( \partial V \).

**EXERCISE (5.8)** Let \( M = \mathbb{C}^2 \). Show that for

\[
    f = \sqrt{z_1(z_1^2 + z_2^3)} \quad , \quad \sqrt{z_1^3 + z_2^5} \quad , \quad \sqrt{z_1(z_2^2 + z_1^n)}
\]

\((n \geq 2) , \quad \sqrt{z_1^2 - z_2^n} \quad (n \geq 2)\) the dual weighted trees obtained from the preceding process are, respectively, \( E_7, E_8, D_{n+1}, A_{n-1} \) (here every tree is weighted by -2).

In general, resolving a singularity of an algebraic function, \( f \), defined on a complex manifold \( M^{(2)} \) amounts to replacing the corresponding singular point in the "Riemann surface" of \( f \) by a finite number of curves, \( K_1, K_2, \ldots, K_n \), \((\dim_{\mathbb{R}} K_i = 2, \text{ for each } i)\), of various genera such that for any pair \( i, j (1 \neq j) \), the curves \( K_i \) and \( K_j \) are either disjoint or intersect at one point regularly. Thus the intersection relations give rise to a dual weighted graph, each vertex corresponds to a curve; furthermore this dual graph is connected. The quadratic form, \( S \), associated with the dual graph (in the same fashion as that associated with a tree)
may be represented by the intersection matrix $M_S = (\alpha_{ij})$, where $\alpha_{ij} = K_i \circ K_j$. We have

**Theorem (5.9)** The quadratic form, $S$, defined above is negative definite.

The proof we are about to give is due to D. Mumford (Institut des Hautes Études Scientifiques, Publications Mathématiques, No. 9, p. 6). Let the multiplicity of $K_i$ be denoted by $m_i$, which is a positive integer for each $i = 1, 2, ..., n$. Define a quadratic form

$$S' \sim \sum_{i,j} \alpha'_{ij} x_i x_j,$$

where $\alpha'_{ij} = m_i m_j \alpha_{ij} = m_i K_i \circ m_j K_j$. It is clear that $S$ is negative definite if, and only if, $S'$ is so.

Notice that

a) $\alpha'_{ij} \geq 0$ if $i \neq j$.

In virtue of (5.2) we have

b) $\sum_{i} \alpha'_{ij} = \sum_{i} (m_i K_i \circ m_j K_j) \leq 0$, for each $i = 1, ..., n$, and

c) $\sum_{i} \alpha'_{ij} < 0$ for some $j$.

Conditions a) and b) imply that $S'$ is negative semi-definite. To obtain the definiteness we write $S'$ in the form

$$\sum_{ij} \alpha'_{ij} x_i x_j = \sum_{i} (\sum_{j} \alpha'_{ij}) x_j^2 - \sum_{i<j} \alpha'_{ij} (x_i - x_j)^2.$$
and put $S' = 0$. Condition c) shows that $x_j = 0$ for some $j$. This together with the connectedness of the dual graph proves that $x_1 = x_2 = \ldots = x_n = 0$, i.e., $S'$ is definite. The proof of (5.9) is thus completed.
6. A Theorem of Kervaire and Milnor

We devote this section to prove Theorem (2.5) which we stated (without proof) in §2. This Theorem, as well as the proof we are about to see, is due to Kervaire and Milnor [2].

Let $M$ be an unbounded (differentiable, oriented, compact) 4-manifold. Let $\tau : H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}_2)$ be the reduction modulo 2 and let $d \in H^2(M;\mathbb{Z})$ be such that $\pi d = w_2(M)$. Under Poincaré duality, $d$ corresponds to an element $b \in H_2(M;\mathbb{Z})$. Theorem (2.5) states that

**THEOREM (2.5)** If $b$ can be represented by a differentiable embedding, $f : S^2 \to M$, then the self-intersection number, $b \circ b = (d \cup d)[M]$, is congruent to $\tau(M)$ modulo 16.

First let us consider the case where $b \circ b = -1$. Then $f(S^2)$ in $M$ has normal bundle equivalent to the bundle $\gamma_{-1}$ defined in §4. In other words $f(S^2)$ has in $M$ a tubular neighborhood, $A$, whose boundary, $\partial A$, considered as the normal $S^1$-bundle of $f(S^2)$ is the Hopf fibration $S^3 \to S^2$. Put $M_1 = (M \setminus \text{Int } A) \cup e^4$, where $e^4$ is a 4-cell whose boundary $\partial e^4 = S^3$, is attached to $\partial A = S^3$ by the identity map. Then $M = M_1 \# (-\mathbb{C}P^2)$. 

Clearly \( w_2(M_1) = 0 \) and hence \( \tau(M_1) \equiv 0 \pmod{16} \) by (2.2). Since \( \tau(M) = \tau(M_1) + \tau(-\mathbb{C}P^2) = \tau(M_1) - 1 \), our theorem is verified for this particular case.

By reversing the orientation of \( M \) if necessary, we may suppose that \( b \cdot b = s \geq 0 \). Let \( P_1, P_2, \ldots, P_{s+1} \) be \( s + 1 \) copies of \(-\mathbb{C}P^2\) and let \( M' = M \# P_1 \# P_2 \# \cdots \# P_{s+1} \).

Using the natural isomorphism

\[
J : H_2(M; \mathbb{Z}) \oplus H_2(P_1; \mathbb{Z}) \oplus \cdots \oplus H_2(P_{s+1}; \mathbb{Z}) \to H_2(M'; \mathbb{Z}),
\]

let

\[
c = J(b \oplus g_1 \oplus \cdots \oplus g_{s+1}),
\]

where \( g_i \) denotes a generator of \( H_2(P_i; \mathbb{Z}) \). We then have

\[
c \cdot c = b \cdot b + \sum_{i=1}^{s+1} g_i \cdot g_i = -1.
\]

Using the hypothesis that \( b \) can be represented by a differentiable embedding of \( S^2 \) in \( M \), it follows easily that \( c \) can be represented by a differentially embedded 2-sphere. Applying the special case we just proved we get

\[
\tau(M') \equiv -1 \pmod{16}.
\]

Since \( \tau(M') = \tau(M) - (s+1) \), we deduce that

\[
b \cdot b = s \equiv (M) \pmod{16}.
\]
EXAMPLE (6.1) Let $M = S^2 \times S^2$ and let $\alpha, \beta \in H^2(M;\mathbb{Z})$ be the standard generators. Then $H^2(M;\mathbb{Z}) = \mathbb{Z} \cdot \alpha \oplus \mathbb{Z} \cdot \beta$ and with respect to this basis the quadratic form, $S_M$, of $M$ is given by the matrix
\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]
An element in $H^2(M;\mathbb{Z})$ is of the form $m\alpha + n\beta$. Its reduction modulo 2, $\tau(m\alpha + n\beta)$, is equal to $w_2(M)$ if, and only if, $m$ and $n$ are both even. Assume this to be the case and let $m = 2m'$, $n = 2n'$. Using the relation
\[(2m'\alpha + 2n'\beta)^2 = 8m'n' \alpha \cup \beta,
\]
it follows that the dual homology class of $m\alpha + n\beta$ can be represented by a differentiably embedded 2-sphere only if $8m'n'$ is divisible by 16 or $m'n' \equiv 0 \pmod{2}$. In particular, $2\alpha + 2\beta$ is not representable by a differentiably embedded 2-sphere in $M$.

EXAMPLE (6.2) Let $M = \mathbb{C}P^2$ and let $g \in H^2(M;\mathbb{Z})$ be the standard generator. The dual homology class of $g$ is obviously represented by a differentiably embedded 2-sphere; however the dual class of $3g$, for example, is not. Indeed, an element, $ng$, of $H^2(M;\mathbb{Z})$ satisfies the relation $\tau(ng) = w_2(M)$ if, and only if, $n$ is odd.
Consider an element \((2k + 1)g\) in \(H^2(M; \mathbb{Z})\). If its dual homology class is representable by a differentiably embedded 2-sphere then \((2k + 1)^2 \equiv 1 \pmod{16}\); i.e. \(k \equiv 0, 3 \pmod{4}\).

On the other hand, the dual homology class of \(3g\) is represented by a combinatorially embedded 2-sphere in \(M = \mathbb{CP}^2\). To see this we let \(z_0, z_1, z_2\) be the homogeneous coordinates in \(\mathbb{CP}^2\) and let \(P = P(z_0, z_1, z_2)\) be a homogeneous polynomial in three variables of (total) degree \(n\). This polynomial defines a divisor of \(\mathbb{CP}^2\) in the obvious fashion (e.g., in the chart where \(z_0 \neq 0\) we consider the function \(P(1, z_1/z_0, z_2/z_0)\)), and the homology class of this divisor is represented by the algebraic curve \(P = 0\).

If \(Q = Q(z_0, z_1, z_2)\) is another homogeneous polynomial of the same degree, \(n\), then \(P/Q\) is a meromorphic function, globally defined on \(\mathbb{CP}^2\); therefore, as a divisor it gives the zero homology class. In other words, the curves \(P = 0\) and \(Q = 0\) represent the same homology class of \(\mathbb{CP}^2\). Since the dual class of \(g\) is represented by the complex projective line \(z_0 = 0\), the homology class represented by the curve \(P = 0\) for any homogeneous polynomial, \(p\), in \(z_0, z_1, z_2\) of degree 3 is dual to \(3g\). In particular we let \(P = z_1^2z_0 - z_2^3\).
The curve $P = 0$ has only one singularity which is a cusp at $(0,0,1)$. Using the formula
\[
\text{genus} = \frac{(n-1)(n-2)}{2} - \text{local terms},
\]
we see that this curve has genus zero. It follows that the curve $P = 0$ considered as a (real) surface in the 4-manifold $\mathbb{CP}^2$ is a 2-sphere (smooth except at a cusp) which is apparently combinatorially embedded.

REMARK (6.3) Kervaire and Milnor proved that in the manifolds $S^2 \times S^2$ and $\mathbb{CP}^2$, any 2-dimensional homology class can be represented by a combinatorially embedded 2-sphere.
7. Integral Unimodular Quadratic Forms

Let \( f = (f, V) \) be a non-degenerate integral quadratic form of rank \( n \). The set of non-negative integers, \( \{ |f(x, x)| \}_{x \in V} \) has a minimum, which we shall denote by \( \min f \). In case \( \min f = 0 \), we say that \( f \) is a zero form or, equivalently, \( f \) represents zero.

If a base \( \{e_i\}_{1 \leq i \leq n} \) is chosen in the lattice, \( V \), the quadratic form, \( f \), may be expressed by

\[
(7.1) \quad f = \sum_{i,j} a_{ij} x_i x_j .
\]

Recall that we can always choose the base \( \{e_i\} \) of \( V \) so that the expression (7.1) of \( f \) is Hermite-reduced. The last term can be defined inductively as follows:

1) If \( f = (f, V) \) is of rank 1, then \( f = a_{11} x_1^2 \) is Hermite reduced.

2) For quadratic form of rank \( n > 1 \), the expression \( f = \sum a_{ij} x_i x_j \) is Hermite-reduced if

\[
\begin{align*}
& a) \quad |a_{11}| \geq 2 |a_{1j}| \quad \text{for} \quad j > 1 , \\
& b) \quad |a_{11}| = \min f , \\
& c) \quad a_{11} f_1 = (\sum a_{1i} x_i)^2 + f_1(x_2, \ldots, x_n) ,
\end{align*}
\]

where \( f_1(x_2, \ldots, x_n) \) is a Hermite-reduced form of rank \( n-1 \).

* The content of this section is based on an article of J-P. Serre (see Serre [1]) as much as on the original lectures.
We have

**THEOREM (7.2)** If $f = \sum a_{ij} x_i x_j$ is Hermite-reduced and if $f$ is not a zero form, then

$$|a_{11}| \leq \frac{4}{3^{(n-1)/2}} |\det f|^{1/n}.$$  

**COROLLARY (7.3)** Let $f = \sum a_{ij} x_i x_j$ be an integral unimodular quadratic form of rank $\leq 5$. If $f$ is not a zero form then it is equivalent to either $\sum y_i^2$ or $-\sum y_i^2$.

This corollary shows in particular that any indefinite unimodular integral quadratic form of rank $\leq 5$ represents zero. For quadratic forms of rank $\geq 5$ this is true for any indefinite form. Precisely, we have

**THEOREM (7.4)** Every non-degenerate indefinite integral quadratic form of rank $\geq 5$ represents zero (Meyer).

Proofs of these statements can be found in the useful monograph of B. W. Jones (Carus Monographs No. 10; see Theorem 23, Corollary 23 and Corollary 27d there).

We now turn to study the Grothendieck-Witt ring $G_0(\mathbb{Z})$ which we defined earlier in Appendix A. An integral unimodular quadratic form, $f$, represents an element (also denoted by $f$) in the monoid $\mathcal{F}_0(\mathbb{Z})$ (see page A-2). The elements in $\mathcal{F}_0(\mathbb{Z})$ represented by the forms $x^2$, $-x^2$ and $xy$ will be denoted respectively by
1, -1 and u. The element in $G_0(Z)$ represented by $f \in \mathcal{F}_0(Z)$ will be denoted by $\overline{f}$. This notation differs slightly from that used in Appendix A.

**Lemma (7.5)** Every unimodular indefinite odd quadratic form of rank $\geq 2$ can be decomposed into the form $1 \dagger (-1) \dagger g$, where $g$ is a non-singular form.

To prove this lemma we let $f = (f, V)$ be a given unimodular indefinite odd form. By (7.3) and (7.4), $f$ is a zero form. Let $x \in V$ be an indivisible element satisfying $f(x, x) = 0$. Since $f$ is non-singular (unimodular), the correlation, $\mathcal{S}$, of $f$ maps $x$ into an indivisible element $\mathcal{S}(x)$ in $V'$. Therefore there exists an element $y \in V$ such that $\langle \mathcal{S}(x), y \rangle = f(x, y) = 1$. We may furthermore choose $y$ so that $f(y, y)$ is odd. For if $f(y, y)$ is even, we choose an element $z \in V$ satisfying $f(z, z) = 1 \mod 2$ (this is possible because $f$ is odd) and replace $y$ by $y' = z + [1 - f(x, y)]y$.

Then $f(y', y')$ is odd and $f(x, y') = 1$. Now assume $f(y, y)$ is odd and let $f(y, y) = 2m + 1$. The elements $e_1 = y - mx$, $e_2 = y - (m+1)x$ are indivisible in $V$ for $f(e_1, e_1) = 1$ and $f(e_2, e_2) = -1$. Our lemma now follows from (0.4).

**Theorem (7.6)** If $f$ is a unimodular indefinite odd quadratic form then $f$ is equivalent to $\alpha^+ \cdot 1 \dagger \alpha^- (-1)$. 
Indeed, Lemma (7.5) shows that \( f \) is equivalent to \( 1 \cdot (-1)^i g \). Since one of \( 1 \cdot g \) and \( (-1)^i g \) has to be indefinite, (7.6) follows from a simple induction.

COROLLARY (7.7) Two unimodular indefinite odd quadratic forms are equivalent if they have the same rank and same index.

We are now ready to show that \( G_0(\mathbb{Z}) \approx \mathbb{Z} \oplus \mathbb{Z} \) as we announced in (A.5). In fact we shall show that

THEOREM (7.8) \( G_0(\mathbb{Z}) = \mathbb{Z} \cdot \bar{1} \oplus \mathbb{Z} \cdot (-1) \).

For each \( f \in \mathcal{F}_0(\mathbb{Z}) \), either \( 1 \cdot f \) or \( (-1)^i f \) is odd and indefinite. Using (7.6) we conclude that \( \bar{f} = \alpha^+ \bar{1} + \alpha^- \cdot (-1) \) in \( G_0(\mathbb{Z}) \) and this shows that \( \bar{1} \) and \( -\bar{1} \) generates \( G_0(\mathbb{Z}) \). Since the homomorphism \((\alpha^+, \alpha^-) : G_0(\mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z} \) maps these generators into the free generators of \( \mathbb{Z} \oplus \mathbb{Z} \), it follows that \( G_0(\mathbb{Z}) = \mathbb{Z} \cdot \bar{1} \oplus \mathbb{Z} \cdot (-1) \).

REMARK (7.9) Corollary (1.10), which states that \( f(\omega, \omega) - \bar{c}(f) \equiv 0 \pmod{8} \) for any integral unimodular form \( f \), now follows easily from (7.8). The map \( h : \mathcal{F}_0(\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z} \) defined by \( h(f) = \frac{f(\omega, \omega) - \bar{c}}{8} \) (reduced modulo 1) is a monoid-homomorphism which induces a homomorphism \( \bar{h} : G_0(\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z} \). Corollary (1.10) is equivalent to saying that \( \bar{h} \) vanishes identically on \( G_0(\mathbb{Z}) \); but this is clear since \( \bar{h} \) vanishes on the generators \( \bar{1} \) and \( -\bar{1} \) of \( G_0(\mathbb{Z}) \).
The structure of integral unimodular indefinite odd forms is completely determined by Theorem (7.6). The structure of unimodular indefinite even forms can also be determined (see Serre [1]). In particular two such forms are equivalent if, and only if, they have the same rank and same index. Combining this with (7.7), we have

**THEOREM (7.10)** Two integral unimodular indefinite quadratic forms are equivalent if, and only if, they have the same rank and same index.

As for definite forms, our knowledge is meagre. We have, for example,

**THEOREM (7.11)** Two integral unimodular quadratic forms of the same rank \( \leq 8 \) are equivalent if they have the same signature and type (i.e., even or odd).

**REMARK (7.12)** The last theorem ceased to be true for forms of rank \( > 8 \). For example, let \( f \) be the quadratic form associated with \( E_8 \) as defined in P. 1-9 and let \( f' = f + 1 \). Then \( f' \) and \( g = 1 + 1 + \ldots + 1 \) (9 times) are of the same rank, same signature and same type. If (7.11) were true for this case, \( f \) and \( g \) would be equivalent. This is impossible for \( f'(x,x) = 1 \) has 2 solutions by \( g(x,x) = 1 \) has 18 solutions.
8. More about Quadratic Forms.

Let $f$ be a quadratic form over a ring $A$. For an element $a \in A$ we shall denote by $af$ the quadratic form defined by $af(x,y) = a.f(x,y)$.

So far we have considered exclusively quadratic forms which are non-degenerate. This is not an essential restriction; for if $f = (f, V)$ is any quadratic form, the kernel, $\text{Rad}(V)$, of the correlation $\varphi : V \rightarrow V'$ of $f$ is a submodule of $V$ and $f$ induces a non-degenerate quadratic form, $\tilde{f}$, on $V/\text{Rad}(V)$. We shall use $\tilde{\det} f$ (resp. $\tilde{\text{DET}} f$) to denote $\det \tilde{f}$ (resp. $\text{DET} \tilde{f}$, cf. p. 0-10) and agree that if $f$ is totally isotropic (i.e. $f(x,y) = 0$ for all $x$ and $y$ in $V$) then $\tilde{\text{DET}} f = 1$.

**THEOREM (8.1)** Every non-degenerate quadratic form, $f$, over $R(p)$ may be decomposed into

$$ f = f_0 + pf_1 + p^2f_2 + \ldots + p^kf_k, $$

where $f_i^\prime (0 \leq i \leq k)$ is a non-singular quadratic form of rank $r_i \geq 0$.

**COROLLARY (8.3)** Every non-degenerate quadratic form over $R(p)$ ($p \neq 2$) decomposes into unary forms.
The corollary follows immediately from (8.1) and (0.5). To prove (8.1) we choose a basis in the lattice, \( V \), on which \( f \) is defined, and express \( f \) by its matrix

\[
M_f = (\alpha_{ij}) \quad (1, j = 1, \ldots, r).
\]

If every \( \alpha_{ij} \) is divisible by \( p \in R(p) \) we shall say that \( M_f \) is divisible by \( p \) and, if this is the case, put \( M_f = p^t M' \), where \( M' = (\alpha'_{ij}) \) is not divisible by \( p \). Notice that \( M' \) may not be invertible.

a). If not all the diagonal entries of \( M' \) are divisible by \( p \), then some of them are units. Applying (0.4), we may split off these entries.

b). If all the diagonal entries of \( M' \) are divisible by \( p \), then there is at least one entry off the diagonal which is a unit and we lose nothing by assuming this entry be \( \alpha'_{12} \); but then the minor

\[
\begin{pmatrix}
\alpha'_{11} & \alpha'_{12} \\
\alpha'_{21} & \alpha'_{22}
\end{pmatrix}
\]

is non-singular and hence splits off.

By repeated application of a) and b) we get

\[ f = p^t (f_t + g) \]

where \( f_t \) is non-singular and the matrix, \( N \), of \( g \) is divisible by \( p \). We write \( N = p^s N' \) such that \( p^t M' \) and repeat the preceding
argument to obtain \( f = p^t f_t + p^{s+t} (f_{s+t} + g') \), where \( f_{s+t} \) is nonsingular and \( g' \) is divisible by \( p \). Our theorem now follows readily from an obvious induction.

Let \( r_1 (0 \leq i \leq k) \) be the rank of \( f_i \) in (8.2). Then \( r_1 \)'s are uniquely determined by the equivalence class of \( f \). Indeed, the cokernel of the correlation \( \rho : V \to V' \) is isomorphic to \( \sum R(p)/\alpha_j \), where \( \alpha_1 \subseteq \cdots \subseteq \alpha_m \nsubseteq R(p) \) are ideals uniquely defined by \( f \), hence by \( f \) (cf. [B - Alg] CH. VII. \#4. Th. 2). In \( R(p) \) the only ideals are of the form \( p^{t} R(p) \). The number of times \( R(p)/p^{t} R(p) \) appears in \( \sum R(p)/\alpha_j \) is clearly equal to \( r_1 \). This gives an invariant definition of the \( r_1 \)'s.

Each \( f_i \) in (8.2) is a non-singular quadratic form. Therefore \( \det f_i \) is a unit. We define

\[
\gamma_i = \begin{cases} 
1, & \text{if } \det f_i \in R(p)^{**}, \\
-1, & \text{if } \det f_i \notin R(p)^{**}.
\end{cases}
\]

In other words, if we identify \( R(p)^{**}/R(p)^{**} \) with \([1, -1]\) then \( \gamma_i = \text{DET} f_i \). As we shall see that the \( \gamma_i \)'s depend only on the equivalence class of \( f \); they are called the Minkowski invariants of \( f \).

We now seek for an invariant definition of these \( \gamma_i \)'s. If this is done, the invariance of \( \gamma_1 \) is automatically established. Let \( f = (f, V) \). We may extend \( f \) to a quadratic form over \( V \otimes F(p) \). The
extended \( f \), formerly denoted by \( f^{F(p)} \), will also be denoted by \( f \) for simplicity. Consider \( V \subset V \otimes F(p) \) and let \( V^+ \) be the "dual" of \( V \) in \( V \otimes F(p) \), i.e., \( V^+ = \{ x \in V \otimes F(p) \mid f(x, y) \in R(p) \text{ for every } y \in V \} \).

Then \( U = V^+ / V \) is a finite abelian \( p \)-group. The quadratic form, \( f \), induces a bilinear pairing
\[
t: U \times U \to \frac{F(p)}{R(p)}.
\]
Filter the group \( U \) by
\[
\emptyset = U_0 \subset U_1 \subset \ldots \subset U,
\]
where \( U_1 = \{ x \in U \mid p^1 x = 0 \} \). It is clear that every element in the quotient group \( W_1 = U_1 / U_{1-1} \) is of order \( p \). We may consider \( W_1 \) as a vector space over \( \mathbb{Z}_p \).

It is also easy to see that the pairing, \( t \), defined above induces a bilinear pairing
\[
f_1^1: W_1 \times W_1 \to \frac{1}{p^1} \frac{R(p)}{R(p)} \approx \mathbb{Z}_p:
\]
Therefore \( f_1^1 \) is a quadratic form over \( \mathbb{Z}_p \) and \( \widetilde{\text{DET}} f_1^1 \in \frac{\mathbb{Z}_p^*}{\mathbb{Z}_p^{**}} \) is defined. Identifying \( \frac{\mathbb{Z}_p^*}{\mathbb{Z}_p^{**}} \) with \( G_2 = \{ 1, -1 \} \), we see that \( \widetilde{\text{DET}} f_1^1 = \pm 1 \). We claim that \( \gamma_1 = \widetilde{\text{DET}} f_1^1 \) (\( i = 1, \ldots, k \)). To show this we let \( f_1 = (f_1, V_1) \) in (8.2). Then \( V = \oplus_{i=0}^{k} V_1 \) and \( V^+ = \sum_{p} \frac{1}{p^1} V_1 \subset V \otimes F(p) \). We compare the definitions of \( \gamma_1 \) and \( \widetilde{\text{DET}} f_1^1 \) and apply (0.17) to conclude that \( \gamma_1 = \widetilde{\text{DET}} f_1^1 \) for \( i = 1, \ldots, k \). The invariance of \( \gamma_0 \) now follows as a consequence.
The following lemma is not hard to prove (see Jones, p. 91).

**Lemma (8.4)** Let \( f \) and \( g \) be non-singular quadratic forms over \( \mathbb{R}(p) \). For \( p \neq 2 \), \( f \) and \( g \) are equivalent if, and only if, they are equivalent over \( \mathbb{F}(p) \). For \( p = 2 \), \( f \) and \( g \) are equivalent if, and only if, they have the same type and are equivalent over \( \mathbb{F}(2) \).

**Theorem (8.5)** Two non-singular quadratic forms over \( \mathbb{R}(p) (p \neq 2) \) are equivalent if they have the same rank and same \( \text{DET} \).

Let \( f \) and \( g \) be two non-singular quadratic forms over \( \mathbb{R}(p) \) such that \( \text{rk} f = \text{rk} g \) and \( \text{DET} f = \text{DET} g \). Since \( p \neq 2 \), \( c_p(f) = c_p(g) = 1 \) in virtue of \( (0.24) \). It follows from \( (A.9) \) that \( f \) and \( g \), when considered as quadratic forms over \( \mathbb{F}(p) \), are equivalent in \( G(\mathbb{F}(p)) \) (i.e., in the notation of \( \mathcal{G} \), \( f = g \)). We then apply \( (A.1) \) and \( (0.8) \) to conclude that \( f \) and \( g \) are equivalent over \( \mathbb{F}(p) \). Our theorem now follows from \( (8.4) \).

**Corollary (8.6)** For non-degenerate quadratic forms over \( \mathbb{R}(p) (p \neq 2) \), the integers \( r_1, y_1 (1 = 0, 1, \ldots, k) \) constitute a complete set of invariants for the equivalence classes.

Returning now to integral quadratic forms. Two such
forms, \( f \) and \( g \), are said to be equivalent over \( R(p) \) if \( f^{R(p)} \sim g^{R(p)} \). We say that \( f \) is semi-equivalent to \( g \) or \( f \) and \( g \) have the same genus if \( f \) is equivalent to \( g \) over \( R(p) \) for \( p = 2, 3, \ldots \infty \). We have

**THEOREM (8.7)** Two integral even quadratic forms, \( f \) and \( g \), with odd determinants are semi-equivalent if they are equivalent over \( R(p) \) for all \( p \neq 2 \) (including \( p = \infty \)).

In virtue of (8.4), we need only to show that \( f \) and \( g \) are equivalent over \( F(2) \). Notice that \( f \sim g \) over \( R(\infty) \) implies \( \text{rk} f = \text{rk} g \). We then apply (0.25) to obtain \( c_2(f) = c_2(g) \), and hence \( \tilde{c}_2(f) = \tilde{c}_2(g) \). Since \( f \) and \( g \) are even forms, the last equality shows that \( \det f = \det g \pmod{8} \); that is \( \text{DET} f = \text{DET} g \) (cf. (0.17) and the proof of (0.18)). It follows from (A.9), (A.1) and (0.8) that \( f \sim g \) over \( F(2) \) and this proves (8.7).

**THEOREM (8.8)** The rank, index and type (even or odd) form a complete system of invariants of the genus of an integral unimodular quadratic form.

This is a consequence of (0.24), (8.4) and (8.5).

Let \( f = (f, V) \) be an integral quadratic form. The invariants \( \gamma_1 \) and \( r_1 \) of \( f^{R(p)} \) will be denoted by
\( \gamma_1(p) \) and \( r_1(p) \), which are invariants of \( f \). Consider \( V < V \otimes Q \) and let \( V^+ = \{ x \in V \otimes Q \mid f(x,y) \in \mathbb{Z} \ \text{for all} \ y \in V \} \). Define \( U = V^+/V \) as before. The quadratic form, \( f \), then induces a bilinear pairing

\[
t : U \times U \rightarrow Q/\mathbb{Z}.
\]

For any odd prime, \( p \), let \( U_1(p) = \{ u \in U \mid p^iu = 0 \} \) and put \( W_1(p) = \frac{U_1(p)}{U_1^{-1}(p)} \). We then obtain a bilinear pairing

\[
W_1(p) \times W_1(p) \rightarrow \mathbb{Z}_p.
\]

It is clear that \( \gamma_1(p), r_1(p) \) \( (i \geq 1) \) of \( f \) can be defined by means of this pairing.

Let \( Y = Y^{4k} \) be a (compact, differentiable, oriented) \( 4k \)-manifold with boundary \( \partial Y = X \). For simplicity we assume that

1) \( Y \) has no torsion,

2) \( Y \) is acyclic in dimensions \( \neq 0 \) or \( 2k \),

3) \( Y \) is a rational homology sphere.

(These assumptions may be weakened.) Consider

\[
\begin{align*}
H_{2k}(Y) & \rightarrow H_{2k}(Y,X) \rightarrow H_{2k-1}(X) \rightarrow 0 \\
& \downarrow H^{2k}(Y) \rightarrow \downarrow \Hom(H_{2k}(Y), \mathbb{Z})
\end{align*}
\]
Since $H_{2k-1}(X)$ is a finite torsion group, if we put
$V = H_{2k}(Y)$ then $\text{Hom}(H_{2k}(Y)\mathbb{Z}) = H_{2k}(Y, X)$ is essentially
$V^*$ and $H_{2k}(X) = U$. The quadratic form, $S_Y$, of $Y$
induces a bilinear pairing

$$L : U \times U \to \mathbb{Q}/\mathbb{Z},$$

which, as informed reader may have already observed,
gives the linking numbers in $X$. Thus $L$, and con-
sequently $\gamma_i(p)$ and $r_i(p) (i \geq 1, p \text{ odd})$, are
invariants of the oriented homotopy type of $X = \partial Y$.
From (8.7) we conclude that if $Y_1$ and $Y_2$ are
4k-manifolds with boundaries satisfying 1) - 3) and if
the quadratic forms $S_1$ and $S_2$ of $Y_1$ and $Y_2$ are
even, with odd determinants, and are equivalent over
$R(\omega) = R$ then $S_1$ and $S_2$ have the same genus provided
$\partial Y_1$ and $\partial Y_2$ have the same oriented homotopy type.
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