6.

The index of an oriented manifold
and the Todd genus of an almost complex manifold

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Introduction. This note gives several applications of the results announced in the notes [1], [2]. In the note [1] we defined the Todd genus $T(M_n)$ of an almost complex manifold as the "genus" belonging to the power series $\frac{-x}{e^{-x} - 1}$. In the note [2] we considered the index $I(M^{4k})$ of an oriented differentiable manifold and proved that $I(M^{4k})$ can be obtained from the Pontrjagin classes by the power series $\frac{\sqrt{z}}{\tgh \sqrt{z}}$. In the case of an almost complex manifold $M_{2k}$ the index $I(M_{2k})$ can be obtained from the Chern classes by the power series $\frac{x}{\tgh x}$. Therefore, we may summarize:

For almost complex manifolds $M_{2k}$

$$T(M_{2k}) \text{ is the } \frac{-x}{e^{-x} - 1} \text{-genus},$$

and

$$I(M_{2k}) \text{ is the } \frac{x}{\tgh x} \text{-genus}$$

("genus" in the sense of note [1], p. 9).

It will turn out that $T$ and $I$ are closely related, the formal reason for this being the equation

$$\frac{x}{\tgh x} + x = \frac{-2x}{e^{-2x} - 1}.$$ 

By definition, $I$ is always an integer. We prove that $2^{n-1} T(M_n)$ is an integer by using this fact and the theorem of Thom [3] that in an oriented differentiable manifold $M^m$ every $(m-2)$-dimensional homology class with integer coefficients can be represented by a subvariety. The question whether $T(M_n)$ is always an integer remains undecided. It seems curious that the prime 2 is the last obstacle towards a positive answer of this question and that, on the other hand, the prime 2 does not play any exceptional role at all for the Todd polynomials themselves (see the formula for the denominators of the Todd polynomials in Lemma 1.5). The fact that $I$ is always an integer allows us to get rid of all odd primes, i.e. one can prove that $2^n T(M_n)$ is always an integer ($2^n$ is the power of 2 contained in the denominator of $T_n$). Since $2^n T(M_n)$ is (mod 2) the class $U^{2n}$ of Wu (which is zero), one gets that $2^{n-1} T(M_n)$ is an integer. For $n \leq 3$, we can prove by a theorem of Rohlin [4] that $T(M_n)$ is always an integer (this result is due to Thom).
The index of an oriented manifold and the Todd genus of an almost complex manifold

The theory of $T$ and $I$ is in a curious state. The index $I$ has by its original definition a topological meaning; the theory of Thom [3] (cobordisme) made it possible to prove that $I(M^{4k})$ can be obtained from the Pontrjagin classes by the power series $\frac{\sqrt{z}}{\tgh \sqrt{z}}$ (resp. from the Chern classes by $\frac{x}{\tgh x}$). The situation of the $T$-theory, however, is very unsatisfactory. We suspect for algebraic manifolds $V_n$ that $T(V_n) = \Pi(V_n)$ (see Introduction of note [1]). Is it possible to give $T$ an intrinsic meaning for almost complex manifolds such that it is automatically an integer? For example: we know that

$$T(M_3) = \frac{c_1 c_2}{24}$$

is always an integer. What is this integer?

All manifolds occurring in this note are differentiable of class $C^\infty$. All submanifolds are $C^\infty$-differentiably imbedded.

Contents

1. Algebraic preliminaries
2. Manifolds $M^{2k}$ with the Poincaré polynomial $1 + t^k + t^{2k}$
3. The function $I$
4. Formal properties of the Todd genus
5. Steenrod’s reduced powers in almost complex manifolds
6. The Todd genus for manifolds of type $(\pi_i)$
7. Fiberings with the flag manifold $U(n)/T^n$ as fibre
8. The Todd genus. Manifolds of type $(\gamma)$

1. Algebraic preliminaries

In the note [1] we defined the multiplicative sequence $K^Q(a_1, a_2, \ldots, a_i)$ belonging to the power series $Q(x) = 1 + \gamma_1 x + \gamma_2 x^2 + \ldots$. Now let

$$C = \sum_{i = 0}^{\infty} c_i x^i \quad \text{and} \quad D = \sum_{i = 0}^{\infty} d_i x^i \quad (c_0 = d_0 = 1)$$

be two power series with indeterminates as coefficients. We may take $C$ as the power series $Q(x)$ and we obtain the multiplicative sequence $K^C(a_1, a_2, \ldots, a_i)$. Here $K^C_j$ is a polynomial of weight $j$ in the $a_i$, the coefficients of which are polynomials of weight $j$ in the $c_i$. We may also take $D$ to be the power series $Q(x)$ and we obtain the multiplicative sequence $K^P(a_1, a_2, \ldots, a_i)$ where $K^P_j$ is of weight $j$ in the $a_i$ and has coefficients which are polynomials of weight $j$ in the $d_i$. The following formula follows immediately from the definitions:

$$K^C_j(d_1, d_2, \ldots, d_j) = K^P_j(c_1, c_2, \ldots, c_j).$$

We use the formula (1) to determine the coefficient of

$$a_{i_1} a_{i_2} \ldots a_{i_r} \left( i_1 \equiv i_2 \equiv \ldots \equiv i_r, \sum_{s=1}^{r} i_s = j \right) \quad \text{in} \quad K^Q.$$

61
Put again $Q(x) = 1 + \gamma_1 x + \gamma_2 x^2 + \ldots$.

For every $m$ we can split up formally, obtaining

\[(2) \quad 1 + \gamma_1 x + \gamma_2 x^2 + \ldots + \gamma_m x^m = (1 + \eta_1 x) (1 + \eta_2 x) \ldots (1 + \eta_m x) .\]

By formula (1) we get

\[\sum_{j=0}^{\infty} K_j^0 = \prod_{i=1}^{m} (1 + a_1 \eta_i x + a_2 \eta_i^2 x^2 + \ldots + a_r \eta_i^r x^r + \ldots) \mod x^{m+1} .\]

We denote the symmetric function with the generating term

\[\eta_1^{i_1} \eta_2^{i_2} \ldots \eta_r^{i_r} \quad (i_1 \geq i_2 \geq \ldots \geq i_r) ,\]

by

\[\sum \eta_1^{i_1} \eta_2^{i_2} \ldots \eta_r^{i_r} ;\]

and infer that the coefficient of $a_1 a_2 \ldots a_r$ in $K_j^0$ is

\[\sum \eta_1^{i_1} \eta_2^{i_2} \ldots \eta_r^{i_r} \quad \text{(for } j \leq m \text{)} .\]

For $0 < j \leq m$ we denote by $s_j$ the sum $\sum_{k=1}^{m} \eta_k^j$ which does not depend on $m$.

We can write $s_j$ as a polynomial of weight $j$ in the $\gamma_i$ (we put here $s_0 = 1$):

\[s_j = s_j (\gamma_1, \gamma_2, \ldots, \gamma_j) .\]

For example

\[s_0 = 1 \]
\[s_1 = \gamma_1 \]
\[s_2 = -2 \gamma_2 + \gamma_1^2 \]
\[s_3 = 3 \gamma_3 - 3 \gamma_2 \gamma_1 + \gamma_1^3 .\]

We have

\[(3) \quad K_j^0(a_1, \ldots, a_j) = s_j (\gamma_1, \gamma_2, \ldots, \gamma_j) a_j + \ldots \text{ composite terms} .\]

Furthermore, the coefficient of $a_j^r$ in $K_j^0$ is equal to the symmetric function in the $\eta_i$ the generating term of which is $\eta_1^{i_1} \eta_2^{i_2} \ldots \eta_r^{i_r} (i_r \leq m)$.

Since the symmetric function with the generating term $\eta_1 \eta_2$ equals $1/2 (s_1^2 - s_2)$, we have

\[(4) \quad K_j^0(0, \ldots, 0, a_j, 0, \ldots, 0, a_{2j}) = s_{2j} a_{2j} + \frac{1}{2} (s_1^2 - s_2) a_j^2 .\]

We note here also the trivial fact that the coefficient of $a_j^r$ in $K_j^0$ is equal to $\gamma_r$.

By a formula of Cauchy we know that

\[(5) \quad \left( \frac{x}{Q(x)} \right)^r Q(x) = \sum_{j=0}^{\infty} (-1)^j s_j x^j .\]

In the note [1] we considered in particular the Todd sequence $T_j$ belonging to the power series $Q(x) = \frac{-x}{e^{-x} - 1}$. For this power series we have

\[\left( \frac{x}{Q(x)} \right)^r Q(x) = \frac{x}{e^x - 1} \quad \text{and} \quad \sum_{j=0}^{\infty} s_j x^j = \frac{-x}{e^x - 1} .\]
The index of an oriented manifold and the Todd genus of an almost complex manifold

Therefore, in the Todd polynomial $T_j$ the coefficient of $c_j$ is equal to that of $c_1^j$. We have

$$\frac{-x}{e^{-x} - 1} = 1 + \frac{x}{2} + \sum_{k=1}^{\infty} \frac{b_{2k}}{(2k)!} x^{2k}$$

where the $b_{2k}$ are the Bernoulli numbers [5] in the even subscript notation. We define as usual $B_k = (-1)^{k-1} b_{2k}$. The $B_k$ are positive rational numbers:

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30},$$

$$B_5 = \frac{5}{66}, \quad B_6 = \frac{691}{2730}, \quad B_7 = \frac{7}{6}, \quad B_8 = \frac{3617}{510}.$$

$T_{2k+1}$ does not contain the term $a_{2k+1}$ for $k \geq 1$.

(6)

$$T_{2k}(a_1, \ldots, a_{2k}) = \frac{b_{2k}}{(2k)!} a_{2k} + \ldots \text{composite terms}.$$  

Remark. We have already stated in the note [1] that $T_{2k+1}$ is divisible by $a_1$, hence does not contain $a_{2k+1}$ for $k \geq 1$ (see 4, (13a) of the present note). In the note [2] (Theorem I') we considered the polynomial $\frac{2^{2k} T_{2k}(a_1, \ldots, a_{2k})}{2^{2k-1}}$ which has integers mod 2 as coefficients. The number $\frac{2^{2k} b_{2k}}{(2k)!}$ is an integer mod 2 which is $\neq 0$ (mod 2), if and only if $k$ is a power of 2. Since $b_{2k}$ contains 2 exactly with the power one in its denominator (Theorem of von Staudt), we have

$$\frac{2^{2k} b_{2k}}{(2k)!} \neq 0 \ (2), \ \text{if and only if } k \text{ is a power of } 2.$$  

This yields

Lemma 1.1. The polynomial $2^j T_j(a_1, \ldots, a_j)$ reduced mod 2 contains the term $a_j$ if and only if $j$ is a power of 2.

Remark. This lemma can be proved more directly by using the power series

$$\frac{-2x}{e^{-2x} - 1} = 1 + x + x^2 + x^4 + \ldots + x^{2^j} + \ldots \mod 2.$$  

We made in the note [2] the convention that we may also consider multiplicative sequences with respect to a power series $Q(z)$, the only difference being that the indeterminate $x$ is replaced by $z$. We note the following trivial

Lemma 1.2. Write, in indeterminates,

$$z = x^2, \quad \sum_{i=0}^{\infty} (-1)^i p_i z^i = \left( \sum_{i=0}^{\infty} c_i x^i \right) \left( \sum_{i=0}^{\infty} (-1)^i c_i x^i \right).$$

Let $Q(z)$ be a power series in $z$. Define $\hat{Q}(x)$ by $\hat{Q}(x) = Q(x^2)$. We have

$$K_{2j}(c_1, \ldots, c_{2j}) = K_j^{Q}(p_1, \ldots, p_j), \quad K_{2j+1}^{\hat{Q}} = 0.$$  

63
In the note [2] we were led to consider the power series
\[ Q(z) = \frac{\sqrt{z}}{\tgh \sqrt{z}} \]
and the corresponding multiplicative sequence \( L_j \). Let \( \bar{L}_j \) be the multiplicative sequence belonging to \( \frac{z}{\tgh z} \). By Lemma 1.2 we have
\[ \bar{L}_{2j+1} = 0, \quad \bar{L}_{2j}(c_1, \ldots, c_{2j}) = L_j(p_1, \ldots, p_j). \]

(7) Since \( \frac{-2x}{e^{-2x} - 1} = x + \frac{x}{\tgh x} \), we obtain
\[ (7a) \quad q' T_{r(q-1)}(c_1, c_2, \ldots, c_{r(q-1)}) \equiv q' \bar{L}_{r(q-1)}(c_1, c_2, \ldots, c_{r(q-1)}) \mod q, \quad (q \text{ odd}). \]

For \( Q(z) = \frac{\sqrt{z}}{\tgh \sqrt{z}} \) we get \( \left( \frac{z}{Q(z)} \right)' Q(z) = \frac{1}{2} + \frac{1}{2} \cdot \frac{2\sqrt{z}}{\sinh 2\sqrt{z}} \); for \( Q(x) = \frac{x}{\tgh x} \) we have \( \frac{x}{Q(x)} \) \( Q(x) = \frac{2x}{\sinh 2x} \). Hence for formulas (3) and (5)
\[ L_j = \left( \frac{2^{2j}(2^{2j-1} - 1)}{(2j)!} B_j \right) p_j + \ldots \text{ composite terms} \]
(8)
\[ \bar{L}_{2j} = (-1)^j \left( \frac{2^{2j+1}(2^{2j-1} - 1)}{(2j)!} B_j \right) c_{2j} + \ldots \text{ composite terms}. \]

We denote by \( \bar{L} \) the (multiplicative) homomorphism which attaches to a power series \( C = \sum_{i=0}^{\infty} c_i x^i \) the power series \( \bar{L}(C) = \sum_{i=0}^{\infty} \bar{L}_i(c_1, \ldots, c_i) x^i \). The following lemma, which we state without proof, is analogous to the Lemma 1.4 in the note [1].

1 Lemma 1.3. Let \( d_1, d_2, \ldots, d_n, \eta \) be indeterminates, \( d_0 = 1 \). If one reduces
\[ \bar{L} \left( \sum_{i=0}^{n} (1 - \eta x)^{n-i} d_i x^i \right) = \sum_{j=0}^{\infty} \bar{L}_j x^j \]
modulo the ideal generated by \( \sum_{i=0}^{n} (-\eta)^{n-i} d_i \), then every \( \bar{L}_j \) can be written uniquely in the form
\[ \bar{L}_j \equiv k_{j-n-1}^{(n)} \eta^{n-1} + k_{j-n-2}^{(n)} \eta^{n-2} + \ldots + k_0^{(n)} \mod \sum_{i=0}^{n} (-\eta)^{n-i} d_i. \]

\( k_j^{(n)} \) is a polynomial of weight \( s \) in the \( d_i \). We have
\[ k_{j-n-1}^{(n)} = 0, \quad \text{if } j \neq n-1 \]
\[ k_0^{(n-1)} = 1, \quad \text{if } n \text{ is odd} \]
\[ k_0^{(n-1)} = 0, \quad \text{if } n \text{ is even}. \]
The index of an oriented manifold and the Todd genus of an almost complex manifold

We state a lemma on the Todd polynomials mod 2 which follows immediately from the congruence

\[ \frac{-2x}{e^{-2x} - 1} = 1 + x + x^2 + x^4 + \ldots + x^{2^i} + \ldots \pmod{2}. \]

**Lemma 1.4.** For \( a_1 = a_3 = a_5 = \ldots = a_{2k-1} = 0 \), we have \( 2^{2k} T^{2k} (a_1, a_2, \ldots, a_{2k}) = 2^k T^k (a_2, a_4, a_6, \ldots, a_{2k}) \pmod{2} \).

Now we give without proof a formula for the denominators of \( T_k \) and \( L_k \).

**Lemma 1.5.** The polynomials \( T_k \) (resp. \( L_k \)) can be written uniquely as polynomials with relatively prime integer coefficients divided by a positive integer \( \mu(T_k) \) (resp. \( \mu(L_k) \)).

\[ \mu(T_k) = \prod q^{\left\lfloor \frac{k}{q-1} \right\rfloor}. \]

The product is over all primes \( q \) with \( 2 \leq q \leq k+1 \).

\[ \mu(L_k) = \prod q^{\left\lfloor \frac{2k}{q-1} \right\rfloor}. \]

The product is over all odd primes \( q \) with \( 3 \leq q \leq 2k+1 \).

\[ 2^{2k} \mu(L_k) = \mu(T_{2k}). \]

We close these algebraic preliminaries with a short list of the polynomials \( T_j (c_1, c_2, \ldots, c_j) \) and \( L_k (p_1, p_2, \ldots, p_k) \):

\[ T_1 = \frac{1}{2} c_1, \quad T_2 = \frac{1}{12} (c_1^2 + c_2), \quad T_3 = \frac{1}{24} c_1 c_2, \]

\[ T_4 = \frac{1}{720} (-c_4 + c_3 c_1 + 3 c_4^2 + 4 c_2 c_3^2 - c_4^3), \]

\[ T_5 = \frac{1}{1440} (3 c_3^2 c_1 - c_2 c_3^2 + c_3 c_3^2 - c_4 c_1), \]

\[ T_6 = \frac{1}{60480} (2c_6 - 2c_5 c_1 - 9c_4 c_2 - 5c_4 c_2^2 - c_3^2 + 11c_3 c_2 c_1 + 5c_3 c_1^2 + 10c_3^2 + 11c_2^2 c_2 - 12c_2 c_4^2 + 2c_7), \]

\[ L_1 = \frac{1}{3} p_1, \quad L_2 = \frac{1}{45} (7p_2 - p_1^3), \quad L_3 = \frac{1}{3^3 \cdot 5 \cdot 7} (62p_3 - 13p_1 p_2 + 2p_1^3), \]

\[ L_4 = \frac{1}{3^4 \cdot 5^2 \cdot 7} (381p_4 - 71p_3 p_1 - 19p_2^2 + 22p_1^3 p_2 - 3p_1^4), \]

\[ L_5 = \frac{1}{3^5 \cdot 5^2 \cdot 7 \cdot 11} (5110p_5 - 919p_4 p_1 - 336p_3 p_2 + 237p_1^3 p_3 + 127p_2^2 - 291p_3 p_2 + 10p_1^4). \]

65
2. Manifolds $M^{2k}$ with Poincaré polynomial $1 + t^k + t^{2k}$

Let $M^{2k}$ be a compact manifold of $2k$ dimensions (not necessarily orientable). For every element $v \in H^k(M^{2k}, \mathbb{Z}_2)$ we have

$$v = U^k v$$

where $U^k = 2^k T_k(w_1, w_2, \ldots, w_k)$ is the class of Wu (note [2], Theorem I'). As an immediate consequence of Lemma 1.1 we have

**Theorem 2.1.** If there exists an element $v \in H^k(M^{2k}, \mathbb{Z}_2)$ with $v \neq 0$, and if the Stiefel-Whitney classes $w_i$ vanish for $i \equiv k \pmod{2}$, then $k$ is a power of 2.

This theorem implies the known fact [6] that manifolds with the $\mathbb{Z}_2$-Poincaré polynomial

$$1 + t^k + t^{2k} = \sum_{r=0}^{2k} \dim H^r(M^{2k}, \mathbb{Z}_2) \cdot t^r$$

can exist only if $k$ is a power of 2. It implies also that manifolds with the (real) Poincaré polynomial

$$1 + t^k + t^{2k} = \sum_{r=0}^{2k} \dim H^r(M^{2k}, \mathbb{R}) \cdot t^r,$$

which do not have torsion [6], can exist only if $k$ is a power of 2 and $k \neq 1$.

**Remark.** A manifold $M^{12}$ (resp. $M^{20}$) with the (real) Poincaré polynomial $1 + t^6 + t^{12}$ (resp. $1 + t^{10} + t^{20}$) cannot exist.

**Proof.** $1 = I(M^{12}) = L_3(p_1, p_2, p_3) = \frac{62 p_3}{3^3 \cdot 5 \cdot 7}$ is impossible. Similar proof for $M^{20}$ by using $L_5$.

3. The function $I$

In the note [2] we defined for a compact oriented manifold $M^m$ with $m = 4k$ the integer $I(M^m)$. For a manifold $M^m$ with $m \equiv 0 \pmod{4}$ we put $I(M^m) = 0$. We recall that $I(M^m) = I(M^m)$ for manifolds $M^m$, $\tilde{M}^m$ which are cobounding (cobordantes).

Thom [3] proved that in an oriented manifold $M^m$ every homology class $h \in H_{m-2}(M^m, \mathbb{Z})$ can be represented by a submanifold $V^{m-2}$. Moreover, two manifolds $V^{m-2}$, $\tilde{V}^{m-2}$ both representing $h$ are cobounding. Hence the index $I(V^{m-2})$ depends only on $h$. Passing to cohomology we have: for every $u \in H^2(M^m, \mathbb{Z})$ there exists a submanifold $V^{m-2}$, the homology class of which is dual to $u$. The index $I(V^{m-2})$ only depends on $u$ and may be denoted by $I(u)$. The number $I(u)$ is always 0 if $m \equiv 2 \pmod{4}$. The mapping $I$ of $H^2(M^{4k+2}, \mathbb{Z})$ into the integers $\mathbb{Z}$ was considered by Thom, who raised the question whether the mapping $I$ is a topological invariant of $M^{4k+2}$. We shall
express $I(u)$ as a polynomial in $u$ and the Pontrjagin classes of $M^{4k+2}$. For convenience, we introduce for an arbitrary oriented manifold $M^m$ the following notation: for $c \in H^\ast(M, \mathbb{Q})$ ($\mathbb{Q}$: rationals, the $\ast$-superscript denotes cohomology ring) let $\chi(c)$ be the $m$-dimensional component of $c$. When no misunderstanding is possible, we shall identify an element of $H^m(M^m, \mathbb{Q})$ with the corresponding rational number.

If $u \in H^2(M^m, \mathbb{Z})$, then

$$
tgh u = \frac{e^u - e^{-u}}{e^u + e^{-u}} = \sum_{r=1}^{\infty} \frac{t_r}{(2r-1)!} u^{2r-1}
$$

is a well-defined element of $H^\ast(M^m, \mathbb{Q})$. The $t_r$ are the so-called tangent coefficients [5],

$$
t_r = \frac{2^{2r} (2^r - 1)}{2^r} B_r,
$$

which are positive integers. We are now able to formulate the

**Theorem 3.1.** For an oriented manifold $M^{4k+2}$ with the Pontrjagin classes $1, p_1, \ldots, p_k$ we have

$$
I(u) = \chi\left( tgh u \cdot \sum_{r=0}^{k} L_r(p_1, \ldots, p_r) \right).
$$

**Proof.** Represent the class $u \in H^2(M^{4k+2}, \mathbb{Z})$ by a subvariety $V^{4k}$, denote by $i$ the imbedding map $V^{4k} \rightarrow M^{4k+2}$. The tangential bundle of $M^{4k+2}$ restricted to $V^{4k}$ splits up into two factors, the tangential $SO(4k)$-bundle of $V^{4k}$, and the normal $SO(2)$-bundle. The sum of all Pontrjagin classes is $i^\ast(1 + u^2)$ for the normal bundle and $i^\ast(1 + p_1 + p_2 + \ldots + p_k)$ for the restricted tangential bundle of $M^{4k+2}$. Since the Pontrjagin classes obey the Whitney duality theorem (modulo torsion), we see that the Pontrjagin classes (modulo torsion) of $V^{4k}$ are given by the formula

$$
\sum_{j=0}^{k} p_j(V^{4k}) = i^\ast[(1 + p_1 + p_2 + \ldots + p_k)(1 + u^2)^{-1}] .
$$

Since $L_j$ is a multiplicative sequence we have

$$
\sum_{j=0}^{k} L_j(p_1(V^{4k}), \ldots, p_j(V^{4k})) = i^\ast\left[ \frac{tgh u}{u} \sum_{j=0}^{k} L_j(p_1, \ldots, p_j) \right].
$$

By Theorem II of the note [2] we know that

$$
I(V^{4k}) = L_k(p_1(V^{4k}), \ldots, p_k(V^{4k})).
$$

This completes the proof.

**Examples.**

$$
tgh u = u - \frac{2}{3!} u^3 + \frac{16}{5!} u^5 - \frac{272}{7!} u^7 + \ldots = u - \frac{1}{3} u^3 + \frac{2}{15} u^5 - \frac{17}{3^2 \cdot 5 \cdot 7} u^7 + \ldots
$$

$M^6$: $I(u) = \frac{1}{3} (-u^3 + p_1 u)$,
\[ M^{10}: I(u) = \frac{1}{45} (6u^5 - 5p_1u^3 + (7p_2-p_1^2)u), \]

\[ M^{14}: I(u) = \frac{1}{3^3 \cdot 5 \cdot 7} (-51u^7 + 42p_1u^5 - (49p_2 - 7p_1^2)u^3 + (62p_3 - 13p_1p_2 + 2p_1^3)u). \]

**Remark.** We obtain for a manifold \( M^6: u^3 \equiv p_1 u \pmod{3} \), for a manifold \( M^{10}: u^5 \equiv -(7p_2-p_1^2)u \pmod{5} \), for a manifold \( M^{14}: u^7 \equiv \frac{1}{2} (62p_3 - 13p_1p_2 + 2p_1^3)u \pmod{7} \).

This agrees with Theorem I of the note [2]. The function \( I(u) \) can be generalized in the following way. Let \( u_1, u_2, \ldots, u_r \left( r \equiv \left\lceil \frac{m}{2} \right\rceil \right) \) be elements of \( H^2(M^m, \mathbb{Z}). \) Represent \( u_1 \) by a submanifold \( V^{m-2} \) of \( M^m, \) restrict \( u_2 \) to \( V^{m-2} \) and represent this restriction by a submanifold \( V^{m-4} \) of \( V^{m-2} \) etc. Finally restrict \( u_r \) to \( V^{m-2(r-1)} \) and represent this restriction by a subvariety \( V^{m-2r} \) of \( V^{m-2(r-1)} \). By successive applications of the formulas (9), Theorem 3.1, we get

**Theorem 3.2.** The index \( I(V^{m-2r}) \) depends only on \( (u_1, u_2, \ldots, u_r) \), without order, and may be denoted by \( I(u_1, u_2, \ldots, u_r) \). We have

\[ I(u_1, u_2, \ldots, u_r) = \pi \left( \tgh u_1 \cdot \tgh u_2 \cdot \ldots \cdot \tgh u_r \cdot \sum_{j=0}^{\infty} L_j(p_1, \ldots, p_j) \right). \]

By using Theorem 3.2 and the formula

\[ \tgh(u+v) = \tgh u + \tgh v - \tgh u \tgh v \tgh (u+v) \]

we obtain

**Theorem 3.3.** For a manifold \( M^{4k+2} \) and \( u, v \in H^2(M^{4k+2}, \mathbb{Z}) \) we have

\[ I(u+v) = I(u) + I(v) - I(u,v,u+v). \]

If \( uv(u+v) = 0 \), then \( I(u+v) = I(u) + I(v) \). In the case of an \( M^6 \), we have that \( I(u,v,u+v) = uv(u+v) \). Now we give a theorem on the index \( I \) which is analogous to the Theorem 4.4 in the note [1].

**Theorem 3.4.** Let \( \mathcal{S} \) be a bundle with the compact oriented manifold \( M^r \) of \( r \) topological dimensions as base, the complex-projective space \( P_{n-1} \) of \( n-1 \) complex dimensions as fibre, and the group of all projective transformations of \( P_{n-1} \) as structure group. We have for the manifold \( \mathcal{S} \)

\[ I(\mathcal{S}) = I(M^r)I(P_{n-1}), \quad \text{i.e.,} \]

\[ I(\mathcal{S}) = I(M^r), \quad \text{if } n \text{ is odd,} \]

\[ I(\mathcal{S}) = 0, \quad \text{if } n \text{ is even.} \]

**Proof.** The theorem is trivial if \( r \) is odd. Therefore, we may assume that \( r = 2m \). The tangential bundle of the manifold \( \mathcal{S} \) "splits up" into two factors, the bundle of
The index of an oriented manifold and the Todd genus of an almost complex manifold
tangential vectors of $\mathcal{T}$ tangential to the fibres and the bundle induced by the
projection from the tangential bundle of the base. The first factor admits
$U(n - 1)$ as structure group. In the notation of Section 3 of note [1] the Chern
class ($= \sum$ of all Chern classes in the various dimensions) of this first factor is
$$
\sum_{i=0}^{n} (1 - \eta)^{n-i} d_i .
$$

We regard the rational cohomology ring of $M^{2m}$ as a subring of the rational
cohomology ring of $\mathcal{T}$ and obtain by using the formula 1.(7)
$$
\sum_{j=0}^{\infty} L_j (p_1 (\mathcal{T}), \ldots, p_j (\mathcal{T})) x^{2j} =
\sum_{j=0}^{\infty} L_j (p_1 (M^{2m}), \ldots, p_j (M^{2m})) x^{2j}.
$$
(10)

(Here $x$ is an indeterminate, the coefficient of $x^j$ is always of complex dimen-
sion $j$). On the right side of the equation (10) every product in the $d_i$ and $p_i$
of complex dimension greater than $m$ vanishes because the $d_i$ and $p_i$ belong to the
cohomology ring of $M^{2m}$. Hence, we get, from Lemma 1.3, that the coefficient of
$x^{m+n-1}$ on the right side of (10) is $(-\eta)^{n-1} L_{m/2} (p_1 (M^{2m}), \ldots)$, if $m$ is even
and $n$ is odd, and is zero otherwise. Therefore, we have for $m$ even and $n$ odd:
$$
L_{1/2 (m+n-1)} (p_1 (\mathcal{T}), \ldots) = (-\eta)^{n-1} L_{m/2} (p_1 (M^{2m}), \ldots).
$$

Since $(-\eta)$ restricted to the fibre $P_{n-1}$ is dual to the positive hyperplane $P_{n-2}$
of $P_{n-1}$, we obtain

$$
I (\mathcal{T}) = I (M^{2m}) , \text{ if } n \text{ is odd and } m \text{ is even},
$$
$$
I (\mathcal{T}) = 0 \text{ otherwise. Q.E.D.}
$$

4. Formal properties of the Todd genus

This section stands in analogy to the preceding section. However, the
results discussed here are almost entirely formal properties of the Todd
polynomials. The function $I$ has a direct topological interpretation and takes
only integral values. But it is still unknown whether the Todd genus coincides
with the $IT$-genus for algebraic manifolds (see Introduction of note [1]) and
whether $T(M_n)$ is always an integer.

Let $M_n$ be a compact almost complex manifold of $n$ complex dimensions
and $u \in H^2 (M_n, \mathbb{Z})$. Denote by $c_i$ the Chern classes of $M_n$. We define
$$
T (u) = n \left[ (1 - e^{-u}) \sum_{j=0}^{n-1} T_j (c_1, \ldots, c_j) \right] , \ \text{i.e.,}
$$
$$
T (u) = (-1)^{n-1} \frac{u^n}{n!} + (-1)^{n-2} \frac{u^{n-1}}{(n-1)!} T_1 (c_1) + \ldots + u T_{n-1} (c_1, \ldots, c_{n-1}) .
$$

Theorem 4.1. Assume that $u \in H^2 (M_n, \mathbb{Z})$ can be represented by an admissible
almost complex subvariety $V_{n-1}$. Denote by $i$ the imbedding map $V_{n-1} \to M_n$.
The Chern classes of $V_{n-1}$ are given by

$$1 + c_1(V_{n-1}) + c_2(V_{n-1}) + \ldots + c_{n-1}(V_{n-1}) = i^* [(1 + c_1 + \ldots + c_n) (1 + u)^{-1}].$$

In particular, we have for the Euler-Poincaré characteristic of $V_{n-1}$ the formula [7]

$$(-1)^{n-1} E(V_{n-1}) = u^n - u^{n-1} c_1 + u^{n-2} c_2 - \ldots + (-1)^{n-1} u c_{n-1}.$$

Moreover, the Todd genus of $V_{n-1}$ equals $T(u)$.

The proof is similar to that of Theorem 3.1. We call $T(u)$ the virtual Todd genus of $u$. The function $T(u)$ can be generalized in the following way. Let $u_1, \ldots, u_r$ be elements of $H^2(M_n, \mathbb{Z})$. We define

$$T(u_1, u_2, \ldots, u_r) = \chi \left[ (1 - e^{-u_1})(1 - e^{-u_2}) \ldots (1 - e^{-u_r}) \sum_{j=0}^{n-r} T_j(c_1, c_2, \ldots, c_j) \right]$$

and have, by successive applications of Theorem 4.1:

**Theorem 4.2.** Assume that $u_1$ can be represented by an admissible almost complex subvariety $V_{n-1}$, that the restriction of $u_2$ to $V_{n-1}$ can be represented by an admissible almost complex subvariety $V_{m-2}$ of $V_{m-1}$, ..., and that finally the restriction of $u_r$ to $V_{m-r+1}$ can be represented by an admissible almost complex subvariety $V_{m-r}$ of $V_{m-r+1}$. Under these assumptions

$$T(V_{m-r}) = T(u_1, u_2, \ldots, u_r).$$

By using the formula

$$1 - e^{-(u+v)} = (1 - e^{-u}) + (1 - e^{-v}) - (1 - e^{-u})(1 - e^{-v})$$

we obtain

**Theorem 4.3.** For an almost complex manifold $M_n$ and $u, v \in H^2(M_n, \mathbb{Z})$ we have

$$T(u + v) = T(u) + T(v) - T(u, v).$$

We define

$$\chi^T(u) = \chi \left( e^u \sum_{j=0}^n T_j(c_1, \ldots, c_j) \right)$$

and have

$$(11) \quad T(M_n) = T(u) + \chi^T(-u)$$

$$(11) \quad T(M_n) = \chi^T(0).$$

Theorem 4.3, formula (11) and other formulas stated at the end of this section are algebraic identities in indeterminates $c_i, u, v, \ldots$ etc. They can be proved easily from the definition of the Todd polynomials in the note [1]. They are interesting because very probably the following formula (2) is true for every non-singular algebraic manifold $M_n$: 70
Let $D$ be a divisor of $M_n$, the homology class of which is dual to the element $d \in H^2(M_n, \mathbb{Z})$. Let $D$ denote also the faisceau of germs of meromorphic functions which are multiples of the divisor $-D$. Define [8]

$$\chi(D) = \sum_{r=0}^{n} (-1)^r \dim H^r(M_n, D).$$

The formula in question is

$$\chi(D) = \chi^T(d).$$

In particular,

$$\chi(0) = \Pi(M_n) = T(M_n).$$

Formulas. Let $M_n$ be almost complex and let $1, c_1, c_2, \ldots$ be the Chern classes of $M_n$. The following formulas are formal algebraic identities. One may regard $c_1, c_2, \ldots, c_n$ as indeterminates and $T(M_n)$ as a symbol for $T_n(c_1, c_2, \ldots, c_n)$.

4 (12) \hspace{1cm} \chi^T(u) = (-1)^n \chi^T(-u - c_1)

(13a) \hspace{1cm} T(c_1) = 2T(M_n), \quad \text{if } n \text{ is odd}

(This yields a proof that $T_n(c_1, \ldots, c_n)$ is divisible by $c_1$ for odd $n$),

(13 b) \hspace{1cm} T(c_1) = 0, \quad \text{if } n \text{ is even}.

We denote by $\psi_j(0 \leq j \leq n)$ the rational number

$$T(-c_1, -c_1, \ldots, -c_1).$$

$j$ times

We have

$$\psi_0 = T(M_n), \quad \psi_n = (-c_1)^n.$$

If $n$ is odd, then we have the formulas

$$\sum_{j=0}^{n} \psi_j = -T(M_n) = \chi^T(-c_1), \quad \sum_{j=1}^{n} \psi_j = -2T(M_n)
$$

(14)

$$\sum_{j=2k}^{n} \left[ \binom{j-k-1}{k-1} + \binom{j-k}{k} \right] \psi_j = 0, \quad 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

If $n$ is even, then we have the formulas

$$\sum_{j=0}^{n} \psi_j = T(M_n) = \chi^T(-c_1), \quad \sum_{j=1}^{n} \psi_j = 0
$$

(15)

$$\sum_{j=2k+1}^{n} \binom{j-k-1}{k} \psi_j = 0, \quad 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Under the assumption that $M_n$ is algebraic and that the Todd genus coincides with the classical arithmetic genus, the numbers $\psi_j$ introduced here are integers and essentially identical with the classical invariants $\Omega_i$, namely

$$\psi_n = \Omega_0, \quad \psi_j = (-1)^{n-j} \Omega_{n-j} + 1 \quad (0 < j < n).$$
The relations (14), (15) are due to E. A. Maxwell and J. A. Todd [9]. As special cases one gets the relations of Segre and Severi [9]. We state one more relation [9] which can be derived from (14). If \( n \) is odd, then

\[
\sum_{j=0}^{n} 2^{n-j} \psi_j = 0. 
\]

**Remark.** In the case that the \( \psi_j \) are integers, this formula implies

\[ c_i^q \equiv 0 \pmod{2}. \]

In fact, we have for an orientable manifold \( M^{4k+2} \) with vanishing Stiefel-Whitney class \( w_3 \) that

\[ 0 = Sq^2 w_2^k = w_2^{2k} U^2 = w_2^{2k+1} \]

(see note [1] Theorem I').

5. Steenrod's reduced powers in almost complex manifolds

Theorem I of the note [2] gives a formula for the classes \( s^q_i \) in terms of Pontrjagin classes. Theorem Y gives a formula for the classes of Wu (Steenrod squares). By using formulas 1,(7) and Lemma 1.4 we can summarize these results for the special case of an almost complex manifold in the following theorem.

**Theorem 5.1.** Let \( M_n \) be almost complex and let \( c_i \in H^{2i}(M_n, \mathbb{Z}) \) be the Chern classes of \( M_n \). For \( k + 2r (q - 1) = 2n \) the class \( s^q_i \in H^{2r(q - 1)}(M_n, \mathbb{Z}) \) is defined by

\[ \mathcal{P}^q v = s^q_i v \quad \text{for all} \quad v \in H^k(M_n, \mathbb{Z}_q). \]

We have the formula

\[ s^q_i \equiv q^r T_r(q - 1)(c_1, c_2, \ldots, c_{r(q - 1)}) \mod q. \]

The classes \( s^q_i \) are obtained from the Chern classes \( \mod q \) by the power series

\[ 1 + (x - x^q + x^{q^2} - x^{q^3} + \ldots)^{q - 1}. \]

For the classes \( U^i \) of Wu (\( U^i \in H^i(M_n, \mathbb{Z}_2) \)) defined by

\[ Sq^i v = U^i v \quad \text{for all} \quad v \in H^{2n-i}(M_n, \mathbb{Z}_2), \]

we have the formula

\[ U^{2r+1} = 0, \quad U^{2r} \equiv 2^r T_r(c_1, \ldots, c_r) \mod 2. \]

The classes \( U^{2r} (r = 1, 2, \ldots, n) \) are obtained from the Chern classes \( \mod 2 \) by the power series \( 1 + x + x^2 + \ldots + x^{2^n} + \ldots \). We define \( s^q_i = U^{2r} \) and have the formula

\[ s^q_i = q^r T_r(q - 1)(c_1, \ldots, c_{r(q - 1)}) \quad \text{for all primes}. \]

We are going to give some applications of Theorem 5.1.
Theorem 5.2. Let $M_n$ be almost complex and $q$ a prime. If $r > \left\lceil \frac{n}{q} \right\rceil$, then
\[ q^r T_{r(q-1)}(c_1, c_2, \ldots, c_{r(q-1)}) \equiv 0 \pmod{q}. \]

Note. These congruences are not trivial (see the Lemma 1.5). Now we give a theorem of Borel-Serre [10] on unitary sphere bundles (in a slightly different formulation) and then the application of this theorem to the tangential bundle of an almost complex manifold. We write $\mathcal{P}_q^r$ instead of $S_q^{2r}$.

Let $[E, X, U(n)]$ be a unitary $(2n-1)$-sphere bundle with the Chern classes $c_i \in H^{2i}(X, \mathbb{Z})$. Write formally
\[ 1 + c_1 x + c_2 x^2 + \ldots + c_n x^n = (1 + u_1 x) (1 + u_2 x) \ldots (1 + u_n x), \]
i.e., $x$ is an indeterminate, the $c_i$ are the elementary symmetric functions in the $u_i$. In the following formulas only symmetric functions in the $u_i$ occur which have to be regarded as polynomials in the $c_i$. A symmetric function will be characterized by its generating term. For example
\[ \sum u_i^2 \cdot u_j^3 = \sum_{i_1, i_2} u_{i_1}^2 \cdot u_{i_2}^3. \]

Theorem 5.3 (Borel-Serre [10]). For the Chern classes $c_1, c_2, \ldots, c_n$ of $[E, X, U(n)]$, we have the following formulas mod $q$ ($q \geq 2$):
\[ \mathcal{P}_q^r c_k = \mathcal{P}_q^r \sum u_1 u_2 \ldots u_k = \sum u_1^q u_2^q \ldots u_{r+1}^q u_{r+1} \ldots u_k (r \leq k). \]
In particular ($q \equiv n + 1$)
(17) \[ \mathcal{P}_q^1 c_{n-q+1} = c_{n-q+1} \sum u_1^{q-1} - c_{n-q+2} \sum u_1^{q-2} + \ldots - c_{n-1} c_1 + n c_n. \]

Remark. In case $q = n+1$, $\mathcal{P}_q^1 c_{n-q+1} = 0$ and (17) is an algebraic identity, namely the familiar recursion formula for the sum $\sum u_i^q$.

5 Corollary 5.3. If for an $[E, X, U(n)]$ the Chern classes $c_1, c_2, \ldots, c_{n-1}$ vanish, then $c_n \equiv 0 \pmod{q}$ for all primes $q \equiv n$ which do not divide $n$.

In the case that $[E, X, U(n)]$ is the tangential bundle of an almost complex manifold $M_n$, we can compare (17) with the formulas ($2 \leq q \leq n + 1$)
\[ \mathcal{P}_q^1 c_{n-q+1} = c_{n-q+1} s_q^1 = c_{n-q+1} \sum u_1^{q-1} \]
and obtain

Theorem 5.3*. For the Chern classes $c_1, c_2, \ldots, c_n$ of $M_n$ we have the following formulas mod $q$ ($2 \leq q \leq n + 1$):
\[ c_{n-q+2} \sum u_1^{q-2} - c_{n-q+3} \sum u_1^{q-3} + \ldots - c_{n-1} c_1 + n c_n \equiv 0 \pmod{q}. \]
For example, $n c_n = 0 \pmod{2}$, $c_{n-1} c_1 - n c_n \equiv 0 \pmod{3},$
\[ c_{n-3} (3 c^3 - 3 c_1 c_2 + c_3^2) - c_{n-2} (c_1^2 - 2 c_2) + c_{n-1} c_1 - n c_n \equiv 0 \pmod{5}. \]
If all the characteristic numbers \(c_{k_1} \cdots c_{k_r} = c_{n \geq i_1 \geq i_2 \geq \ldots \geq i_r + i_1 + i_2 + \ldots + i_r = n}\) vanish, then \(c_n\) (= the Euler-Poincaré characteristic of \(M_n\)) is divisible by all primes \(q \leq n + 1\) which do not divide \(n\).

**Corollary 5.3*. If in an almost complex manifold \(M_n\) the Chern classes \(c_1, c_2, \ldots, c_{\left\lfloor \frac{n}{2} \right\rfloor}\) vanish, then the Euler-Poincaré characteristic of \(M_n\) is divisible by all primes \(q \leq n + 1\) which do not divide \(n\).

**Remark.** Corollary 5.3* implies that the spheres \(S^{2k}\), except \(S^2, S^6\), are not almost complex (Borel-Serre, only \(S^4\) escapes Corollary 5.3). It seems to be interesting to ask for sharper theorems on the Euler-Poincaré characteristic of an almost complex manifold \(M_n\) for which all characteristic numbers

\[c_{k_1} \cdots c_{k_r} (n - 1 \geq i_1 \geq i_2 \geq \ldots \geq i_r, i_1 + i_2 + \ldots + i_r = n)\]

vanish. In the last section we shall return to this problem. For which complex dimensions \(n\) does there exist an almost complex manifold \(M_n\) for which these characteristic numbers vanish and for which \(c_n \neq 0\)? For which complex dimensions \(n\) does there exist an almost complex manifold with \(c_1 = c_2 = \ldots = \left\lfloor \frac{n}{2} \right\rfloor = 0\) and \(c_n \neq 0\)? (The algebraic surface of degree 4 in \(P_3\) supplies an example to the second question for \(n = 2\). In this example \(c_1 = 0\) and \(c_2 = 24\).)

**Remark.** By combining Theorems 5.1 and 5.3 one can obtain numerous congruences for the Chern classes for an almost complex manifold \(M_n\).

### 6. The Todd genus for manifolds of type \((\pi_r)\)

In Sections 6 and 8 we study properties of the Todd genus of an almost complex manifold \(M_n\). In particular, we prove in Section 8 that \(2^{n-1} T(M_n)\) is always an integer. In Section 7 we make some general preparations for Section 8.

We say that an almost complex manifold \(M_n\) of \(n\) complex dimensions is of type \((\pi_r)\), if there exist elements \(u_1, u_2, \ldots, u_r \in H^2(M_n, \mathbb{Z})\) such that

\[1 + c_1 + c_2 + \ldots + c_n = (1 + u_1)(1 + u_2)\ldots(1 + u_r)\]

\((c_i\) denotes the Chern classes of \(M_n\), \(c_i \in H^{2i}(M_n, \mathbb{Z})\)). For example, the complex projective space \(P_n\) is of type \((\pi_{n+1})\). If \(M_n\) is of type \((\pi_r)\), we say simply "\(M_n\) is of type \((\pi)\)". For a manifold \(M_n\) of type \((\pi_r)\) we can express the Todd genus by means of the function \(I\).

**Theorem 6.1.** Let \(M_n\) be of type \((\pi_r)\) and

\[1 + c_1 + c_2 + \ldots + c_n = (1 + u_1)(1 + u_2)\ldots(1 + u_r).\]
The index of an oriented manifold and the Todd genus of an almost complex manifold

We have

\[ 2^n T(M_n) = I(M_n) + \sum_{i=1}^{r} I(u_i) + \sum_{1 \leq i_1 < i_2 \leq r} I(u_{i_1}, u_{i_2}) + \ldots + \sum_{1 \leq i_1 < \ldots < i_r \leq r} I(u_{i_1}, u_{i_2}, \ldots, u_{i_r}) , \]

\[ 2^n T(M_n) = I(M_n) + \sum_{j=1}^{r} \sum_{1 \leq i_1 < \ldots < i_s \leq r} I(u_{i_1}, u_{i_2}, \ldots, u_{i_s}) . \]

**Proof.** (\(\pi\) denotes the 2\(n\)-dimensional component.) We have the following formulas:

\[ \frac{-2x}{e^{-2x} - 1} = \frac{x}{\tgh x} + x , \]

\[ 2^n T(M_n) = \frac{-u_1}{e^{-u_1} - 1} \cdot \frac{-u_2}{e^{-u_2} - 1} \ldots \frac{-u_r}{e^{-u_r} - 1} = \prod_{i=1}^{r} \left( \frac{u_i + \frac{u_i}{\tgh u_i}}{\tgh u_i} \right) , \]

\[ \sum_{j=0}^{\infty} L_j (p_1(M_n), \ldots) = \sum_{j=0}^{\infty} \hat{L}_{2j} (c_1(M_n), \ldots) = \prod_{i=1}^{r} \frac{u_i}{\tgh u_i} . \]

(See Lemma 1.2.)

These formulas together with Theorem 3.2 establish the proof.

Theorem 6.1 yields the

**Lemma 6.2.** The number \(2^n T(M_n)\) is an integer for every almost complex manifold of type \((\pi_r)\).

7

**7. Fiberings with the flag manifold** \(U(n)/T^n\) **as fibre**

We shall prove that \(2^{n-1} T(M_n)\) is always an integer by constructing, for every \(M_n\), an almost complex manifold (of higher dimension) which is of type \((\pi)\) and the Todd genus of which coincides with \(T(M_n)\). Before doing so we have to make some general preparations. (See Borel [12] and Chern [13].)

For a group \(G\) and a subgroup \(U\) we define \(G/U\) by "\(g_1 \sim g_2 \iff g_1^{-1} g_2 \in U\)" \((g_1, g_2 \in G)\). The group \(G\) regarded as group of left translations of \(G\) operates on \(G/U\).

Let \(U(n)\) be the unitary group regarded as group of all unitary \(n \times n\) matrices. The group \(U(n_1) \times U(n_2) \times \ldots \times U(n_k)\) with \(n_1 + n_2 + \ldots + n_k = n\) is a subgroup of \(U(n)\)

\[
\begin{pmatrix}
U(n_1) \\
U(n_2) \\
\vdots \\
U(n_k)
\end{pmatrix} \subseteq U(n).
\]

For \(U(1) \times \ldots \times U(1) \times U(n-k)\) we write \(T^k \times U(n-k)\). In particular, \(T^n\) is the subgroup of all diagonal matrices (maximal torus of \(U(n)\)). By
\( T^{k-1} \times 1 \times U(n-k) \) we denote the subgroup of \( T^k \times U(n-k) \) consisting of those matrices which have 1 at the \( k^\text{th} \) place of the diagonal.

The manifold \( U(n)/T^n \) is the so-called flag manifold (= space of all \( n \) frames of unitary orthogonal complex line elements in the origin of the \( n \) dimensional vector space over the complex numbers). The manifold \( U(n)/T^n \) is almost complex (\( U(n)/T^n \) is even a rational algebraic manifold). The manifold \( U(n)/T^n \) can be fibred in complex projective lines \( P_1 \):

\[
U(n)/T^n \xrightarrow{P_k} U(n)/T^{n-2} \times U(2).
\]

Moreover, we get a sequence of fiberings in complex projective spaces

\[
U(n)/T^{n-2} \times U(2) \xrightarrow{P_2} U(n)/T^{n-3} \times U(3)
\]

\[
U(n)/T^{n-3} \times U(3) \xrightarrow{P_3} U(n)/T^{n-4} \times U(4)
\]

\[
\vdots
\]

\[
U(n)/T^1 \times U(n-1) \xrightarrow{P_{n-1}} U(n)/U(n).
\]

The bundle

\[ (18) \quad U(n)/T^k \times U(n-k) \xrightarrow{P_{n-k}} U(n)/T^{k-1} \times U(n-k+1) \quad (1 \leq k \leq n) \]

admits \( U(n-k+1) \) as structure group with the associated bundle

\[ (19) \quad U(n)/T^{k-1} \times 1 \times U(n-k) \xrightarrow{S^{k-1}} U(n)/T^{k-1} \times U(n-k+1). \]

We also have the fiberings \((1 \leq k \leq n)\)

\[ (20) \quad U(n)/T^{k-1} \times 1 \times U(n-k) \xrightarrow{S^1} U(n)/T^k \times U(n-k). \]

The manifolds \( U(n)/T^k \times U(n-k), 0 \leq k \leq n, \) occurring in the fiberings (18) have a natural almost complex structure compatible with the fiberings. The group \( U(n) \) operates on all manifolds occurring in (18), (19), (20) in such a way that these fiberings are preserved. Therefore, if \([E, X, U(n)]\) is a unitary \((2n-1)\)-sphere bundle, we may construct the associated bundles \( \mathcal{F}_k \rightarrow X \) and \( \mathcal{F}_k \rightarrow X \) with \( U(n)/T^k \times U(n-k) \) (resp. \( U(n)/T^{k-1} \times 1 \times U(n-k) \)) as fibre \((1 \leq k \leq n)\), and we get a sequence of fiberings which we illustrate by the following commutative diagram.

\[
(21)
\]

We denote by \( u_r \), the characteristic class of the \( S^1 \)-bundle \( \mathcal{F}_r \rightarrow \mathcal{F}_r \) \((u_r \in H^2(\mathcal{F}_r, \mathbb{Z}))\). The cohomology ring \( H^\ast(\mathcal{F}_r, \mathbb{Z}) \) may be regarded as a subring of \( H^\ast(\mathcal{F}_{r+1}, \mathbb{Z}) \). We observe that the bundle \( \mathcal{F}_{r+1} \rightarrow \mathcal{F}_r \), lifted up to \( \mathcal{F}_{r+1} \) splits up over \( \mathcal{F}_{r+1} \) in the bundles \( \mathcal{F}_{r+1} \rightarrow \mathcal{F}_{r+1} \) and \( \mathcal{F}_{r+2} \rightarrow \mathcal{F}_{r+1} \). By successive applications of the Whitney duality theorem for Chern classes we obtain by induction over \( k \) \( (k = n-r, k = 1, 2, \ldots, n) \).

**Lemma 7.1.** The Chern class of the bundle

\[
\mathcal{F}_{n-k+1} \xrightarrow{S^{k-1}} \mathcal{F}_{n-k} \text{ is } (1 + u_{n-k+1})(1 + u_{n-k+2}) \ldots (1 + u_n).
\]

76
The index of an oriented manifold and the Todd genus of an almost complex manifold

Now we can apply Theorem 3.1 (Remark 2) of the note [1] to the complex-projective bundle

\[ \mathcal{S}_{n-1} \xrightarrow{p_{k-1}} \mathcal{S}_n. \]

That is, we consider the space \( V(P_{k-1}) \) of all unit tangential vectors of \( P_{k-1} \)

\[ V(P_{k-1}) \xrightarrow{S^{2k-3}} P_{k-1} \]

and the associated bundle \( \mathcal{S}_{n-k+1, V} \) with fibre \( V(P_{k-1}) \)

\[ \mathcal{S}_{n-k+1, V} \xrightarrow{V(P_{k-1})} \mathcal{S}_{n-k} \]

\[ \xrightarrow{S^{2k-3}} P_{k-1} \]

\[ \mathcal{S}_{n-k+1} \xrightarrow{S^1} \mathcal{S}_{n-k+1} \]

As indicated in the diagram, \( \mathcal{S}_{n-k+1, V} \) is a unitary \((2k-3)\)-sphere bundle over \( \mathcal{S}_{n-k+1} \). By Theorem 3.1 of the note [1] we infer that the Chern class of this bundle is given by

\[ c(\mathcal{S}_{n-k+1, V}, \mathcal{S}_{n-k+1}, U(k-1)) = \prod_{j=n-k+1}^{n} (1 + u_j - u_{n-k+1}). \]  

(23)

Repeated application of formula (23) leads to the

**Lemma 7.2.** Let \( M_r \) be an almost complex manifold and denote by \( 1 + c_1 + c_2 + \ldots + c_r \) the Chern class of its tangential bundle. Let

\[ E \xrightarrow{S^{2n-1}} M_r, \]

be a unitary \((2n-1)\)-sphere bundle over \( M_r \), with the Chern class \( 1 + d_1 + d_2 + \ldots + d_n \). Make the construction of the diagram (21) with respect to the bundle \( E \). Then we arrive at a manifold \( \mathcal{S}_{n-1} \) of complex dimension

\[ r + \frac{n(n-1)}{2} \]

which has a natural almost complex structure compatible with the fiberings of the diagram (21):

\[ \mathcal{S}_{n-1} \xrightarrow{U(n)/T^n} M_r. \]

We have the formulas \( u_i \in H^2(\mathcal{S}_{n-1}, \mathbb{Z}) \) and \( c_i, d_i \in H^{2i}(M_r, \mathbb{Z}) \):

\[ 1 + d_1 + d_2 + \ldots + d_n = (1 + u_1) (1 + u_2) \ldots (1 + u_n). \]

The Chern class of the tangential bundle of \( \mathcal{S}_{n-1} \) is

\[ (1 + c_1 + c_2 + \ldots + c_r) \cdot \prod_{i>j} (1 + u_i - u_j). \]

Remark. The lemma applies in particular to the case that \( M_r \) is a point. We obtain a formula for the Chern class of the tangential bundle of the manifold \( U(n)/T^n \)

\[ c(U(n)/T^n) = \prod_{i>j} (1 + u_i - u_j). \]  

(24)
F. Hirzebruch

Hence \( U(n)/T^n \) is of type \((\pi)\). We have

\[
u_i \in H^2(U(n)/T^n, \mathbb{Z})
\]

and

\[(25) \quad (1 + u_1)(1 + u_2) \cdots (1 + u_n) = 1.\]

The cohomology ring \( H^*(U(n)/T^n, \mathbb{Z}) \) is generated by the \( u_i (1 \leq i \leq n) \) with the only relation (25) (Borel [12]). By the method employed in this section the Chern class of the manifold

\[
U(n)/(U(n_1) \times U(n_2) \times \cdots \times U(n_k)), \quad n_1 + n_2 + \cdots + n_k = n,
\]

can be determined. Moreover, A. Borel pointed out to the author that (24) can be generalized to an arbitrary compact Lie group \( G \): Let \( T \) be a maximal torus. The manifold \( G/T \) is a complex manifold of type \((\pi)\). Namely, if \( r_1, \ldots, r_m \in H^2(G/T, \mathbb{Z}) \) are a choice of “positive roots” of \( G \), then \( G/T \) has an almost complex structure with \( c(G/T) = \prod_{i=1}^{m} (1 + r_i) \), \( m = \) complex dimension of \( G/T \). In particular, the product of all roots is the Euler-Poincaré characteristic of \( G/T \).

8. The Todd genus. Manifolds of type \((\gamma)\)

We are now able to prove the

**Theorem 8.1.** The number \( 2n-1 T(M_n) \) is an integer for every compact almost complex manifold.

**Proof.** We construct the almost complex manifold \( \mathcal{A}_{n-1} \) of Lemma 7.2 taking as bundle \( E \) the tangential bundle of \( M_n \) (hence \( r = n \) and \( c_i = d_i \)). For the Chern class of \( \mathcal{A}_{n-1} \) we have the formula

\[
c(\mathcal{A}_{n-1}) = (1 + u_1)(1 + u_2) \cdots (1 + u_n) \prod_{i \geq j} (1 + u_i - u_j).
\]

Therefore, the manifold \( \mathcal{A}_{n-1} \) is of type \((\pi)\). We know from Theorem 4.4 of the note [1] that

\[
T(M_n) = T(\mathcal{A}_{n-1}).
\]

By Lemma 6.2 the Todd genus \( T(M_n) \) multiplied with a power of 2 is an integer. The Todd polynomial \( T_n \) is a polynomial with (relatively prime) integer coefficients divided by a big denominator (Lemma 1.5). This denominator contains 2 exactly \( n \) times. Hence \( 2^n T(M_n) \) is an integer which by Theorem 5.2 is even. This completes the proof.

No example of an almost complex manifold \( M_n \) is known where \( T(M_n) \) is not an integer. R. Thom has communicated to the author that \( T(M_2) \) and \( T(M_3) \) are always integers. He pointed out that this fact is related to a recent theorem of Rohlin [13].

*If the second Stiefel-Whitney class of an orientable compact differentiable manifold vanishes, then the Pontrjagin class \( p_1(V^n) \) is divisible by 48.*
The index of an oriented manifold and the Todd genus of an almost complex manifold

Before showing how "\( T(M_3) \) is an integer" can be derived from this theorem of Rohlin we state the following trivial lemma:

**Lemma 8.2.** If the statement "\( T(M_n) \) is an integer for all complex manifolds of complex dimension \( n \)" is true for \( n = n_0 \), then it is true for all \( n < n_0 \).

**Proof.** Assume \( n < n_0 \). We have

\[
T(M_n) = T(M_n) \times T(P_{n_0-n}) = T(M_n \times P_{n_0-n}) = \text{integer}.
\]

**Theorem 8.3 (Thom).** The Todd genus is an integer for all almost complex manifolds \( M_2, M_3 \).

**Proof.** By Lemma 8.2 it is sufficient to prove that \( T(M_3) \) is an integer. Represent in \( M_2 \) the Chern class \( c_1 \) by a subvariety \( V^4 \) (not necessarily almost complex). By Whitney duality the second Stiefel-Whitney class of \( V^4 \) is 0. By 3.(9a) we find that

\[
p_1(V^4) = -2c_1c_2.
\]

By Rohlin \(-2c_1c_2 \equiv 0 \) (48). Q.E.D.

We are now returning to the question discussed in Theorem 5.3* (Remark). For brevity we say that an almost complex manifold \( M_n \) is of type \((\gamma)\) when all characteristic numbers \( c_{i_1}c_{i_2}\ldots c_{i_r} \) with \( n-1 \geq i_1 \geq i_2 \geq \ldots \geq i_r \) and \( i_1 + i_2 + \ldots + i_r = n \) vanish. We say that \( M_n \) is of type \((\gamma')\) when

\[
c_1, c_2, \ldots, c[\frac{n}{2}]
\]

vanish. Obviously \((\gamma')\) implies \((\gamma)\). For a manifold \( M_{2k} \) of type \((\gamma)\) the Todd genus and the index are rational multiples of the Euler-Poincaré characteristic \( c_{2k} \) of \( M_{2k} \). By 1,(6) and 1,(8) we obtain the formulas \((B_k, B(k)\) Bernoulli number):

\[
I(M_{2k}) = (-1)^k \frac{2^{2k+1}(2^{2k-1}-1)}{(2k)!} B_k \cdot c_{2k},
\]

\[
T(M_{2k}) = (-1)^{k-1} \frac{B_k}{(2k)!} \cdot c_{2k},
\]

\[
I(M_{2k}) = -2^{2k+1}(2^{2k-1}-1) T(M_{2k}).
\]

Because \( 2^{2k-1} T(M_{2k}) \) is an integer we obtain by (27) divisibility properties for \( c_{2k} \). The question whether these divisibility properties formally imply those of Theorem 5.3* seems to be of a difficult arithmetic nature involving curious properties of the Bernoulli numbers \( B_k \). Since \( B_k \) is always \( \neq 0 \), we can see that for a manifold \( M_{2k} \) of type \((\gamma)\) either \( T, I, c_{2k} \) are all \( 0 \) or \( \neq 0 \). The formulas (26), (27), (28) are of a rather questionable character, because of missing examples (see the two questions, 5.3* Remark).
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Footnotes


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