6.

The index of an oriented manifold and the Todd genus of an almost complex manifold

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Introduction. This note gives several applications of the results announced in the notes [1], [2]. In the note [1] we defined the Todd genus $T(M_n)$ of an almost complex manifold as the "genus" belonging to the power series $\frac{-x}{e^{-x}-1}$. In the note [2] we considered the index $I(M^{4k})$ of an oriented differentiable manifold and proved that $I(M^{4k})$ can be obtained from the Pontrjagin classes by the power series $\frac{\sqrt{z}}{\operatorname{tgh}\sqrt{z}}$. In the case of an almost complex manifold M_{2k} the index $I(M_{2k})$ can be obtained from the Chern classes by the power series $\frac{x}{\operatorname{tgh} x}$. Therefore, we may summarize:

For almost complex manifolds M_{2k}

$$T(M_{2k}) \text{ is the } \frac{-x}{e^{-x}-1}\text{-genus},$$

and

$$I(M_{2k}) \text{ is the } \frac{x}{\operatorname{tgh} x}\text{-genus}$$

("genus" in the sense of note [1], p. 9).

It will turn out that T and I are closely related, the formal reason for this being the equation

$$\frac{x}{\operatorname{tgh} x}+x=\frac{-2x}{e^{-2x}-1}.$$

By definition, I is always an integer. We prove that $2^{n-1}\,T(M_n)$ is an integer by using this fact and the theorem of Thom [3] that in an oriented differentiable manifold M^m every $(m-2)$-dimensional homology class with integer coefficients can be represented by a subvariety. The question whether $T(M_n)$ is always an integer remains undecided. It seems curious that the prime 2 is the last obstacle towards a positive answer of this question and that, on the other hand, the prime 2 does not play any exceptional role at all for the Todd polynomials themselves (see the formula for the denominators of the Todd polynomials in Lemma 1.5). The fact that I is always an integer allows us to get rid of all odd primes, i.e. one can prove that $2^n\,T(M_n)$ is always an integer (2^n is the power of 2 contained in the denominator of T_n). Since $2^n\,T(M_n)$ is (mod 2) the class U^{2n} of Wu (which is zero), one gets that $2^{n-1}\,T(M_n)$ is an integer. For $n \leqq 3$, we can prove by a theorem of Rohlin [4] that $T(M_n)$ is always an integer (this result is due to Thom).

The theory of T and I is in a curious state. The index I has by its original definition a topological meaning; the theory of Thom [3] (cobordisme) made it possible to prove that $I(M^{4k})$ can be obtained from the Pontrjagin classes by the power series $\frac{\sqrt{z}}{\operatorname{tgh}\sqrt{z}}$ $\left(\text{resp. from the Chern classes by } \frac{x}{\operatorname{tgh} x}\right)$. The situation of the T-theory, however, is very unsatisfactory. We suspect for algebraic manifolds V_n that $T(V_n) = \Pi(V_n)$ (see Introduction of note [1]). Is it possible to give T an intrinsic meaning for almost complex manifolds such that it is automatically an integer? For example: we know that

$$T(M_3) = \frac{c_1 c_2}{24}$$

is always an integer. What is this integer?

All manifolds occurring in this note are differentiable of class C^∞. All submanifolds are C^∞-differentiably imbedded.

Contents

1. Algebraic preliminaries

In the note [1] we defined the multiplicative sequence $K_j^Q(a_1, a_2, \ldots, a_j)$ belonging to the power series $Q(x) = 1 + \gamma_1 x + \gamma_2 x^2 + \ldots$. Now let

$$C = \sum_{i=0}^{\infty} c_i x^i \quad \text{and} \quad D = \sum_{i=0}^{\infty} d_i x^i \quad (c_0 = d_0 = 1)$$

be two power series with *indeterminates* as coefficients. We may take C as the power series $Q(x)$ and we obtain the multiplicative sequence $K_j^C(a_1, a_2, \ldots, a_j)$. Here K_j^C is a polynomial of weight j in the a_i, the coefficients of which are polynomials of weight j in the c_i. We may also take D to be the power series $Q(x)$ and we obtain the multiplicative sequence $K_j^D(a_1, a_2, \ldots, a_j)$ where K_j^D is of weight j in the a_i and has coefficients which are polynomials of weight j in the d_i. The following formula follows immediately from the definitions:

$$(1) \qquad K_j^C(d_1, d_2, \ldots, d_j) = K_j^D(c_1, c_2, \ldots, c_j)\,.$$

We use the formula (1) to determine the coefficient of

$$a_{i_1} a_{i_2} \ldots a_{i_r} \left(i_1 \geqq i_2 \geqq \ldots \geqq i_r, \sum_{s=1}^{r} i_s = j\right) \quad \text{in } K_j^Q\,.$$

Put again $Q(x) = 1 + \gamma_1 x + \gamma_2 x^2 + \dots$.

For every m we can split up formally, obtaining

(2) $$1 + \gamma_1 x + \gamma_2 x^2 + \dots + \gamma_m x^m = (1 + \eta_1 x)(1 + \eta_2 x) \dots (1 + \eta_m x).$$

By formula (1) we get

$$\sum_{j=0}^{\infty} K_j^Q = \prod_{i=1}^{m} (1 + a_1 \eta_i x + a_2 \eta_i^2 x^2 + \dots + a_r \eta_i^r x^r + \dots) \bmod x^{m+1}.$$

We denote the symmetric function with the generating term

$$\eta_1^{i_1} \eta_2^{i_2} \dots \eta_r^{i_r} \quad (i_1 \geqq i_2 \geqq \dots \geqq i_r),$$

by

$$\sum \eta_1^{i_1} \eta_2^{i_2} \dots \eta_r^{i_r}$$

and infer that the coefficient of $a_{i_1} a_{i_2} \dots a_{i_r}$ in K_j^Q is

$$\sum \eta_1^{i_1} \eta_2^{i_2} \dots \eta_r^{i_r} \quad (\text{for } j \leqq m).$$

For $0 < j \leqq m$ we denote by s_j the sum $\sum_{k=1}^{m} \eta_k^j$ which does not depend on m. We can write s_j as a polynomial of weight j in the γ_i (we put here $s_0 = 1$):

$$s_j = s_j(\gamma_1, \gamma_2, \dots, \gamma_j).$$

For example

$$\begin{aligned} s_0 &= 1 \\ s_1 &= \gamma_1 \\ s_2 &= -2\gamma_2 + \gamma_1^2 \\ s_3 &= 3\gamma_3 - 3\gamma_2\gamma_1 + \gamma_1^3. \end{aligned}$$

We have

(3) $$K_j^Q(a_1, \dots, a_j) = s_j(\gamma_1, \gamma_2, \dots, \gamma_j)\, a_j + \dots \text{ composite terms}.$$

Furthermore, the coefficient of a_j^r in K_{rj}^Q is equal to the symmetric function in the η_i the generating term of which is $\eta_1^j \eta_2^j \dots \eta_r^j$ $(j r \leqq m)$.

Since the symmetric function with the generating term $\eta_1^j \eta_2^j$ equals $\frac{1}{2}(s_j^2 - s_{2j})$, we have

(4) $$K_{2j}^Q(0, \dots, 0, a_j, 0, \dots, 0, a_{2j}) = s_{2j} a_{2j} + \tfrac{1}{2}(s_j^2 - s_{2j})\, a_j^2.$$

We note here also the trivial fact that the coefficient of a_1^r in K_r^Q is equal to γ_r.

By a formula of Cauchy we know that

(5) $$\left(\frac{x}{Q(x)}\right)' Q(x) = \sum_{j=0}^{\infty} (-1)^j s_j x^j.$$

In the note [1] we considered in particular the Todd sequence T_j belonging to the power series $Q(x) = \dfrac{-x}{e^{-x} - 1}$. For this power series we have

$$\left(\frac{x}{Q(x)}\right)' Q(x) = \frac{x}{e^x - 1} \quad \text{and} \quad \sum_{j=0}^{\infty} s_j x^j = \frac{-x}{e^{-x} - 1}.$$

Therefore, in the Todd polynomial T_j the coefficient of c_j is equal to that of c_1^j.
We have

$$\frac{-x}{e^{-x}-1}=1+\frac{x}{2}+\sum_{k=1}^{\infty}\frac{b_{2k}}{(2k)!}x^{2k}$$

where the b_{2k} are the Bernoulli numbers [5] in the even subscript notation. We define as usual $B_k=(-1)^{k-1}b_{2k}$. The B_k are positive rational numbers:

$$B_1=\frac{1}{6},\quad B_2=\frac{1}{30},\quad B_3=\frac{1}{42},\quad B_4=\frac{1}{30},$$

$$B_5=\frac{5}{66},\quad B_6=\frac{691}{2730},\quad B_7=\frac{7}{6},\quad B_8=\frac{3617}{510}.$$

(6) T_{2k+1} does not contain the term a_{2k+1} for $k\geqq 1$.

$$T_{2k}(a_1,\ldots,a_{2k})=\frac{b_{2k}}{(2k)!}a_{2k}+\ldots\text{ composite terms}.$$

Remark. We have already stated in the note [1] that T_{2k+1} is divisible by a_1, hence does not contain a_{2k+1} for $k\geqq 1$ (see 4, (13a) of the present note). In the note [2] (Theorem I′) we considered the polynomial $2^{2k}T_{2k}(a_1,\ldots,a_{2k})$ which has integers mod 2 as coefficients. The number $\frac{2^{2k-1}}{(2k)!}$ is an integer mod 2 which is $\neq 0$ (mod 2), if and only if k is a power of 2. Since b_{2k} contains 2 exactly with the power one in its denominator (Theorem of von Staudt), we have

$$\frac{2^{2k}b_{2k}}{(2k)!}\neq 0\ (2),\ \text{if and only if } k \text{ is a power of } 2.$$

This yields

Lemma 1.1. *The polynomial $2^jT_j(a_1,\ldots,a_j)$ reduced* mod 2 *contains the term a_j if and only if j is a power of* 2.

Remark. This lemma can be proved more directly by using the power series

$$\frac{-2x}{e^{-2x}-1}=1+x+x^2+x^4+\ldots+x^{2^i}+\ldots \bmod 2.$$

We made in the note [2] the convention that we may also consider multiplicative sequences with respect to a power series $Q(z)$, the only difference being that the indeterminate x is replaced by z. We note the following trivial

Lemma 1.2. *Write, in indeterminates,*

$$z=x^2,\quad \sum_{i=0}^{\infty}(-1)^i p_i z^i=\left(\sum_{i=0}^{\infty}c_i x^i\right)\cdot\left(\sum_{i=0}^{\infty}(-1)^i c_i x^i\right).$$

Let $Q(z)$ be a power series in z. Define $\tilde{Q}(x)$ by $\tilde{Q}(x)=Q(x^2)$. We have

$$K_{2j}^{\tilde{Q}}(c_1,\ldots,c_{2j})=K_j^{Q}(p_1,\ldots,p_j),\quad K_{2j+1}^{\tilde{Q}}=0.$$

In the note [2] we were led to consider the power series

$$Q(z)=\frac{\sqrt{z}}{\operatorname{tgh}\sqrt{z}}$$

and the corresponding multiplicative sequence L_j. Let $\tilde{L}_j$ be the multiplicative sequence belonging to $\dfrac{x}{\operatorname{tgh} x}$. By Lemma 1.2 we have

$$\tilde{L}_{2j+1}=0\,, \qquad \tilde{L}_{2j}(c_1,\ldots,c_{2j})=L_j(p_1,\ldots,p_j)\,. \tag{7}$$

Since $\dfrac{-2x}{e^{-2x}-1}=x+\dfrac{x}{\operatorname{tgh} x}$, we obtain

$$q^r\,T_{r(q-1)}(c_1,c_2,\ldots,c_{r(q-1)}) \equiv q^r\,\tilde{L}_{r(q-1)}(c_1,c_2,\ldots,c_{r(q-1)}) \bmod q, \quad (q \text{ odd}). \tag{7a}$$

For $Q(z)=\dfrac{\sqrt{z}}{\operatorname{tgh}\sqrt{z}}$ we get $\left(\dfrac{z}{Q(z)}\right)' Q(z)=\frac{1}{2}+\frac{1}{2}\cdot\dfrac{2\sqrt{z}}{\sinh 2\sqrt{z}}$; for $Q(x)=\dfrac{x}{\operatorname{tgh} x}$ we have $\left(\dfrac{x}{Q(x)}\right)' Q(x)=\dfrac{2x}{\sinh 2x}$. Hence for formulas (3) and (5)

$$\begin{aligned} L_j &= \left(\frac{2^{2j}(2^{2j-1}-1)}{(2j)!}\,B_j\right) p_j+\ldots \text{ composite terms} \\ \tilde{L}_{2j} &= (-1)^j\left(\frac{2^{2j+1}(2^{2j-1}-1)}{(2j)!}\,B_j\right) c_{2j}+\ldots \text{ composite terms.} \end{aligned} \tag{8}$$

We denote by $\tilde{L}$ the (multiplicative) homomorphism which attaches to a power series $C=\sum_{i=0}^{\infty} c_i x^i$ the power series $\tilde{L}(C)=\sum_{i=0}^{\infty}\tilde{L}_i(c_1,\ldots,c_i)\,x^i$. The following lemma, which we state without proof, is analogous to the Lemma 1.4 in the note [1].

1 **Lemma 1.3.** *Let $d_1, d_2, \ldots, d_n, \eta$ be indeterminates, $d_0=1$. If one reduces*

$$\tilde{L}\left(\sum_{i=0}^{n}(1-\eta x)^{n-i} d_i x^i\right)=\sum_{j=0}^{\infty}\tilde{L}_j x^j$$

modulo the ideal generated by $\sum_{i=0}^{n}(-\eta)^{n-i} d_i$, then every $\tilde{L}_j$ can be written uniquely in the form

$$\tilde{L}_j \equiv k^{(j)}_{j-n+1}\,\eta^{n-1}+k^{(j)}_{j-n+2}\,\eta^{n-2}+\ldots+k^{(j)}_j \bmod \sum_{i=0}^{n}(-\eta)^{n-i} d_i\,.$$

$k^{(j)}_s$ is a polynomial of weight s in the d_i. We have

$$\begin{aligned} k^{(j)}_{j-n+1} &= 0\,, && \text{if } j \neq n-1 \\ k^{(n-1)}_0 &= 1\,, && \text{if } n \text{ is odd} \\ k^{(n-1)}_0 &= 0\,, && \text{if } n \text{ is even.} \end{aligned}$$

We state a lemma on the Todd polynomials mod 2 which follows immediately from the congruence

$$\frac{-2x}{e^{-2x}-1} = 1 + x + x^2 + x^4 + \ldots + x^{2^i} + \ldots \pmod 2 .$$

Lemma 1.4. *For* $a_1 = a_3 = a_5 = \ldots = a_{2k-1} = 0$, *we have* $2^{2k} T^{2k}(a_1, a_2, \ldots, a_{2k}) = 2^k T^k(a_2, a_4, a_6, \ldots, a_{2k}) \bmod 2$.

Now we give without proof a formula for the denominators of T_k and L_k.

Lemma 1.5. *The polynomials* T_k (resp. L_k) *can be written uniquely as polynomials with relatively prime integer coefficients divided by a positive integer* $\mu(T_k)$ (resp. $\mu(L_k)$).

$$\mu(T_k) = \prod q^{\left[\frac{k}{q-1}\right]} .$$

The product is over all primes q *with* $2 \leqq q \leqq k+1$.

$$\mu(L_k) = \prod q^{\left[\frac{2k}{q-1}\right]} .$$

The product is over all odd *primes* q *with* $3 \leqq q \leqq 2k+1$.

$$2^{2k} \mu(L_k) = \mu(T_{2k}) .$$

We close these algebraic preliminaries with a short list of the polynomials $T_j(c_1, c_2, \ldots, c_j)$ and $L_k(p_1, p_2, \ldots, p_k)$:

$$T_1 = \frac{1}{2} c_1 , \quad T_2 = \frac{1}{12}(c_1^2 + c_2) , \quad T_3 = \frac{1}{24} c_1 c_2 ,$$

$$T_4 = \frac{1}{720}(-c_4 + c_3 c_1 + 3c_2^2 + 4c_2 c_1^2 - c_1^4) ,$$

$$T_5 = \frac{1}{1440}(3c_2^2 c_1 - c_2 c_1^3 + c_3 c_1^2 - c_4 c_1) ,$$

$$T_6 = \frac{1}{60\,480}(2c_6 - 2c_5 c_1 - 9c_4 c_2 - 5c_4 c_1^2 - c_3^2 + 11 c_3 c_2 c_1 + 5 c_3 c_1^3 + 10 c_2^3 + 11 c_2^2 c_1^2 - 12 c_2 c_1^4 + 2 c_1^6) ,$$

$$L_1 = \frac{1}{3} p_1 , \quad L_2 = \frac{1}{45}(7p_2 - p_1^2) , \quad L_3 = \frac{1}{3^3 \cdot 5 \cdot 7}(62 p_3 - 13 p_1 p_2 + 2 p_1^3) ,$$

$$L_4 = \frac{1}{3^4 \cdot 5^2 \cdot 7}(381 p_4 - 71 p_3 p_1 - 19 p_2^2 + 22 p_1^2 p_2 - 3 p_1^4) ,$$

$$L_5 = \frac{1}{3^5 \cdot 5^2 \cdot 7 \cdot 11}(5110 p_5 - 919 p_4 p_1 - 336 p_3 p_2 + 237 p_1^2 p_3 + 127 p_1 p_2^2 - 83 p_1^3 p_2 + 10 p_1^5) .$$

2 2. Manifolds M^{2k} with Poincaré polynomial $1+t^k+t^{2k}$

Let M^{2k} be a compact manifold of $2k$ dimensions (not necessarily orientable). For every element $v \in H^k(M^{2k}, \mathbf{Z}_2)$ we have

$$v\,v = U^k\,v$$

where $U^k = 2^k T_k(w_1, w_2, \dots, w_k)$ is the class of Wu (note [2], Theorem I′). As an immediate consequence of Lemma 1.1 we have

Theorem 2.1. *If there exists an element $v \in H^k(M^{2k}, \mathbf{Z}_2)$ with $v\,v \neq 0$, and if the Stiefel-Whitney classes w_i vanish for $i \leqq \left[\frac{k}{2}\right]$, then k is a power of 2.*

This theorem implies the known fact [6] that manifolds with the $\mathbf{Z}_2$-Poincaré polynomial

$$1+t^k+t^{2k} = \sum_{r=0}^{2k} \dim H^r(M^{2k}, \mathbf{Z}_2) \cdot t^r$$

can exist only if k is a power of 2. It implies also that manifolds with the (real) Poincaré polynomial

$$1+t^k+t^{2k} = \sum_{r=0}^{2k} \dim H^r(M^{2k}, \mathbf{R}) \cdot t^r,$$

which do not have torsion [6], can exist only if k is a power of 2 and $k \neq 1$.

Remark. A manifold M^{12} (resp. M^{20}) with the (real) Poincaré polynomial $1+t^6+t^{12}$ (resp. $1+t^{10}+t^{20}$) cannot exist.

Proof. $1 = I(M^{12}) = L_3(p_1, p_2, p_3) = \dfrac{62\,p_3}{3^3 \cdot 5 \cdot 7}$ is impossible. Similar proof for M^{20} by using L_5.

3. The function I

In the note [2] we defined for a compact oriented manifold M^m with $m = 4k$ the integer $I(M^m)$. For a manifold M^m with $m \not\equiv 0 \pmod 4$ we put $I(M^m) = 0$. We recall that $I(M^m) = I(\tilde{M}^m)$ for manifolds M^m, $\tilde{M}^m$ which are cobounding (cobordantes).

Thom [3] proved that in an oriented manifold M^m every homology class $h \in H_{m-2}(M^m, \mathbf{Z})$ can be represented by a submanifold V^{m-2}. Moreover, two manifolds V^{m-2}, $\tilde{V}^{m-2}$ both representing h are cobounding. Hence the index $I(V^{m-2})$ depends only on h. Passing to cohomology we have: for every $u \in H^2(M^m, \mathbf{Z})$ there exists a submanifold V^{m-2}, the homology class of which is dual to u. The index $I(V^{m-2})$ only depends on u and may be denoted by $I(u)$. The number $I(u)$ is always 0 if $m \not\equiv 2 \pmod 4$. The mapping I of
3 $H^2(M^{4k+2}, \mathbf{Z})$ into the integers $\mathbf{Z}$ was considered by Thom, who raised the question whether the mapping I is a topological invariant of M^{4k+2}. We shall

express $I(u)$ as a polynomial in u and the Pontrjagin classes of M^{4k+2}. For convenience, we introduce for an arbitrary oriented manifold M^m the following notation: for $c \in H^*(M,\mathbb{Q})$ ($\mathbb{Q}$: rationals, the $*$-superscript denotes cohomology ring) let $\varkappa(c)$ be the m-dimensional component of c. When no misunderstanding is possible, we shall identify an element of $H^m(M^m,\mathbb{Q})$ with the corresponding rational number.

If $u \in H^2(M^m,\mathbb{Z})$, then

$$\operatorname{tgh} u = \frac{e^u - e^{-u}}{e^u + e^{-u}} = \sum_{r=1}^{\infty} (-1)^{r-1} \frac{t_r}{(2r-1)!} u^{2r-1}$$

is a well-defined element of $H^*(M^m,\mathbb{Q})$. The t_r are the so-called tangent coefficients [5],

$$t_r = \frac{2^{2r}(2^{2r}-1)}{2r} B_r,$$

which are positive integers. We are now able to formulate the

Theorem 3.1. *For an oriented manifold M^{4k+2} with the Pontrjagin classes $1, p_1, \ldots, p_k$ we have*

$$I(u) = \varkappa\left(\operatorname{tgh} u \cdot \sum_{r=0}^{k} L_r(p_1, \ldots, p_r)\right).$$

Proof. Represent the class $u \in H^2(M^{4k+2},\mathbb{Z})$ by a subvariety V^{4k}, denote by i the imbedding map $V^{4k} \to M^{4k+2}$. The tangential bundle of M^{4k+2} restricted to V^{4k} splits up into two factors, the tangential $SO(4k)$-bundle of V^{4k}, and the normal $SO(2)$-bundle. The sum of all Pontrjagin classes is $i^*(1+u^2)$ for the normal bundle and $i^*(1+p_1+p_2+\ldots+p_k)$ for the restricted tangential bundle of M^{4k+2}. Since the Pontrjagin classes obey the Whitney duality theorem (modulo torsion), we see that the Pontrjagin classes (modulo torsion) of V^{4k} are given by the formula

$$\text{(9 a)} \qquad \sum_{i=0}^{k} p_i(V^{4k}) = i^*[(1+p_1+p_2+\ldots+p_k)(1+u^2)^{-1}].$$

Since L_j is a multiplicative sequence we have

$$\text{(9 b)} \qquad \sum_{j=0}^{k} L_j(p_1(V^{4k}), \ldots, p_j(V^{4k})) = i^*\left[\frac{\operatorname{tgh} u}{u} \sum_{j=0}^{k} L_j(p_1, \ldots, p_j)\right].$$

By Theorem II of the note [2] we know that

$$I(V^{4k}) = L_k(p_1(V^{4k}), \ldots, p_k(V^{4k})).$$

This completes the proof.

Examples.

$$\operatorname{tgh} u = u - \frac{2}{3!}u^3 + \frac{16}{5!}u^5 - \frac{272}{7!}u^7 + \ldots = u - \frac{1}{3}u^3 + \frac{2}{15}u^5 - \frac{17}{3^2 \cdot 5 \cdot 7}u^7 + \ldots$$

$$M^6\colon\; I(u) = \frac{1}{3}(-u^3 + p_1 u),$$

$$M^{10}\colon\ I(u) = \frac{1}{45}\,(6u^5 - 5p_1 u^3 + (7p_2 - p_1^2)\,u)\,,$$

$$M^{14}\colon\ I(u) = \frac{1}{3^3\cdot 5\cdot 7}\,(-51u^7 + 42p_1 u^5 - (49p_2 - 7p_1^2)\,u^3 + (62p_3 - 13p_1p_2 + 2p_1^3)\,u)\,.$$

Remark. We obtain for a manifold M^6: $u^3 \equiv p_1 u \pmod 3$, for a manifold M^{10}: $u^5 \equiv -(7p_2 - p_1^2)\,u \pmod 5$, for a manifold

$$M^{14}\colon\ u^7 \equiv \tfrac{1}{2}(62p_3 - 13p_1p_2 + 2p_1^3)\,u \pmod 7\,.$$

This agrees with Theorem I of the note [2]. The function $I(u)$ can be generalized in the following way. Let $u_1, u_2, \dots, u_r \left(r \leqq \left[\frac{m}{2}\right]\right)$ be elements of $H^2(M^m, \mathbf{Z})$. Represent u_1 by a submanifold V^{m-2} of M^m, restrict u_2 to V^{m-2} and represent this restriction by a submanifold V^{m-4} of V^{m-2} ... etc. Finally restrict u_r to $V^{m-2(r-1)}$ and represent this restriction by a subvariety V^{m-2r} of $V^{m-2(r-1)}$. By successive applications of the formulas (9), Theorem 3.1, we get

Theorem 3.2. *The index $I(V^{m-2r})$ depends only on $(u_1, u_2, \dots, u_r)$, without order, and may be denoted by $I(u_1, u_2, \dots, u_r)$. We have*

$$I(u_1, u_2, \dots, u_r) = \varkappa\left(\operatorname{tgh} u_1 \cdot \operatorname{tgh} u_2 \cdot \ldots \cdot \operatorname{tgh} u_r \cdot \sum_{j=0}^{\infty} L_j(p_1, \dots, p_j)\right).$$

By using Theorem 3.2 and the formula

$$\operatorname{tgh}(u+v) = \operatorname{tgh} u + \operatorname{tgh} v - \operatorname{tgh} u \operatorname{tgh} v \operatorname{tgh}(u+v)$$

we obtain

Theorem 3.3. *For a manifold M^{4k+2} and $u, v \in H^2(M^{4k+2}, \mathbf{Z})$ we have*

$$I(u+v) = I(u) + I(v) - I(u, v, u+v)\,.$$

If $u\,v\,(u+v) = 0$, then $I(u+v) = I(u) + I(v)$. In the case of an M^6, we have that $I(u, v, u+v) = u\,v\,(u+v)$. Now we give a theorem on the index I which is analogous to the Theorem 4.4 in the note [1].

Theorem 3.4. *Let $\mathscr{S}$ be a bundle with the compact oriented manifold M^r of r topological dimensions as base, the complex-projective space P_{n-1} of $n-1$ complex dimensions as fibre, and the group of all projective transformations of P_{n-1} as structure group. We have for the manifold $\mathscr{S}$*

$$I(\mathscr{S}) = I(M^r)\, I(P_{n-1}),\ \text{i.e.,}$$

$$I(\mathscr{S}) = I(M^r)\,, \qquad \text{if } n \text{ is odd,}$$

$$I(\mathscr{S}) = 0\,, \qquad \text{if } n \text{ is even.}$$

Proof. The theorem is trivial if r is odd. Therefore, we may assume that $r = 2m$. The tangential bundle of the manifold $\mathscr{S}$ "splits up" into two factors, the bundle of tangential vectors of $\mathscr{S}$ tangential to the fibres and the bundle induced by the projection from the tangential bundle of the base. The first factor admits $U(n-1)$ as structure group. In the notation of Section 3 of note [1] the Chern class (= sum of all Chern classes in the various dimensions) of this first factor is

$$\sum_{i=0}^{n} (1-\eta)^{n-i} d_i .$$

We regard the rational cohomology ring of M^{2m} as a subring of the rational cohomology ring of $\mathscr{S}$ and obtain by using the formula 1,(7)

$$(10)\qquad \begin{aligned} &\sum_{j=0}^{\infty} L_j(p_1(\mathscr{S}),\ldots,p_j(\mathscr{S}))\, x^{2j} = \\ &\tilde{L}\left(\sum_{i=0}^{n} (1-\eta x)^{n-i} d_i x^i\right) \cdot \sum_{j=0}^{\infty} L_j(p_1(M^{2m}),\ldots,p_j(M^{2m}))\, x^{2j} . \end{aligned}$$

(Here x is an indeterminate, the coefficient of x^i is always of complex dimension i). On the right side of the equation (10) every product in the d_i and p_i of complex dimension greater than m vanishes because the d_i and p_i belong to the cohomology ring of M^{2m}. Hence, we get, from Lemma 1.3, that the coefficient of x^{m+n-1} on the right side of (10) is $(-\eta)^{n-1} L_{m/2}(p_1(M^{2m}),\ldots)$, if m is even and n is odd, and is zero otherwise. Therefore, we have for m even and n odd:

$$L_{1/2(m+n-1)}(p_1(\mathscr{S}),\ldots) = (-\eta)^{n-1} L_{m/2}(p_1(M^{2m}),\ldots) .$$

Since $(-\eta)$ restricted to the fibre P_{n-1} is dual to the positive hyperplane P_{n-2} of P_{n-1}, we obtain

$$I(\mathscr{S}) = I(M^{2m}) , \quad \text{if } n \text{ is odd and } m \text{ is even,}$$

$$I(\mathscr{S}) = 0 \ \text{ otherwise.} \quad \text{Q.E.D.}$$

4. Formal properties of the Todd genus

This section stands in analogy to the preceding section. However, the results discussed here are almost entirely *formal* properties of the Todd polynomials. The function I has a direct topological interpretation and takes only integral values. But it is still unknown whether the Todd genus coincides with the Π-genus for algebraic manifolds (see Introduction of note [1]) and whether $T(M_n)$ is always an integer.

Let M_n be a compact almost complex manifold of n complex dimensions and $u \in H^2(M_n, \mathbb{Z})$. Denote by c_i the Chern classes of M_n. We define

$$T(u) = \varkappa\left[(1-e^{-u}) \sum_{j=0}^{n-1} T_j(c_1,\ldots,c_j)\right], \quad \text{i.e.,}$$

$$T(u) = (-1)^{n-1}\frac{u^n}{n!} + (-1)^{n-2}\frac{u^{n-1}}{(n-1)!} T_1(c_1) + \ldots + u\, T_{n-1}(c_1,\ldots,c_{n-1}) .$$

Theorem 4.1. *Assume that $u \in H^2(M_n, \mathbb{Z})$ can be represented by an admissible almost complex subvariety V_{n-1}. Denote by i the imbedding map $V_{n-1} \to M_n$.*

The Chern classes of V_{n-1} are given by

$$1+c_1(V_{n-1})+c_2(V_{n-1})+\ldots+c_{n-1}(V_{n-1})=i^*[(1+c_1+\ldots+c_n)(1+u)^{-1}].$$

In particular, we have for the Euler-Poincaré characteristic of V_{n-1} the formula [7]

$$(-1)^{n-1}E(V_{n-1})=u^n-u^{n-1}c_1+u^{n-2}c_2-\ldots+(-1)^{n-1}u\,c_{n-1}.$$

Moreover, the Todd genus of V_{n-1} equals $T(u)$.

The proof is similar to that of Theorem 3.1. We call $T(u)$ the virtual Todd genus of u. The function $T(u)$ can be generalized in the following way. Let $u_1,\ldots,u_r$ $(r\leqq n)$ be elements of $H^2(M_n,\mathbb{Z})$. We define

$$T(u_1,u_2,\ldots,u_r)=\varkappa\left[(1-e^{-u_1})(1-e^{-u_2})\ldots(1-e^{-u_r})\sum_{j=0}^{n-r}T_j(c_1,c_2,\ldots,c_j)\right]$$

and have, by successive applications of Theorem 4.1:

Theorem 4.2. *Assume that u_1 can be represented by an admissible almost complex subvariety V_{m-1}, that the restriction of u_2 to V_{m-1} can be represented by an admissible almost complex subvariety V_{m-2} of V_{m-1}, ..., and that finally the restriction of u_r to V_{m-r+1} can be represented by an admissible almost complex subvariety V_{m-r} of V_{m-r+1}. Under these assumptions*

$$T(V_{m-r})=T(u_1,u_2,\ldots,u_r).$$

By using the formula

$$1-e^{-(u+v)}=(1-e^{-u})+(1-e^{-v})-(1-e^{-u})(1-e^{-v})$$

we obtain

Theorem 4.3. *For an almost complex manifold M_n and $u,v\in H^2(M_n,\mathbb{Z})$ we have*

$$T(u+v)=T(u)+T(v)-T(u,v).$$

We define

$$\chi^T(u)=\varkappa\left(e^u\sum_{j=0}^{n}T_j(c_1,\ldots,c_j)\right)$$

and have

$$T(M_n)=T(u)+\chi^T(-u) \qquad (11)$$
$$T(M_n)=\chi^T(0).$$

Theorem 4.3, formula (11) and other formulas stated at the end of this section are algebraic identities in indeterminates $c_i, u, v, \ldots$ etc. They can be proved easily from the definition of the Todd polynomials in the note [1]. They are interesting because very probably the following formula (?) is true for every non-singular algebraic manifold M_n:

Let D be a divisor of M_n, the homology class of which is dual to the element $d \in H^2(M_n, \mathbb{Z})$. Let D denote also the faisceau of germs of meromorphic functions which are multiples of the divisor $-D$. Define [8]

$$\chi(D) = \sum_{r=0}^{n} (-1)^r \dim H^r(M_n, D)\,.$$

The formula in question is

4 (?) $$\chi(D) = \chi^T(d)\,.$$

In particular,

(?) $$\chi(0) = \Pi(M_n) = T(M_n)\,.$$

Formulas. Let M_n be almost complex and let $1, c_1, c_2, \ldots$ be the Chern classes of M_n. The following formulas are formal algebraic identities. One may regard $c_1, c_2, \ldots, u$ as indeterminates and $T(M_n)$ as a symbol for $T_n(c_1, c_2, \ldots, c_n)$.

4 (12) $$\chi^T(u) = (-1)^n \chi^T(-u - c_1)$$

(13 a) $$T(c_1) = 2\,T(M_n)\,, \quad \text{if } n \text{ is odd}$$

(This yields a proof that $T_n(c_1, \ldots, c_n)$ is divisible by c_1 for odd n),

(13 b) $$T(c_1) = 0\,, \quad \text{if } n \text{ is even}\,.$$

We denote by $\psi_j\,(0 \leqq j \leqq n)$ the rational number

$$T(\underbrace{-c_1, -c_1, \ldots, -c_1}_{j \text{ times}}).$$

We have $$\psi_0 = T(M_n)\,, \qquad \psi_n = (-c_1)^n\,.$$

If n is odd, then we have the formulas

(14)
$$\sum_{j=0}^{n} \psi_j = -T(M_n) = \chi^T(-c_1)\,, \qquad \sum_{j=1}^{n} \psi_j = -2\,T(M_n)$$
$$\sum_{j=2k}^{n} \left[\binom{j-k-1}{k-1} + \binom{j-k}{k}\right] \psi_j = 0\,, \qquad 1 \leqq k \leqq \frac{n-1}{2}\,.$$

If n is even, then we have the formulas

(15)
$$\sum_{j=0}^{n} \psi_j = T(M_n) = \chi^T(-c_1)\,, \qquad \sum_{j=1}^{n} \psi_j = 0$$
$$\sum_{j=2k+1}^{n} \binom{j-k-1}{k} \psi_j = 0\,, \qquad 1 \leqq k \leqq \frac{n}{2} - 1\,.$$

Under the assumption that M_n is algebraic and that the Todd genus coincides with the classical arithmetic genus, the numbers ψ_j introduced here are integers and essentially identical with the classical invariants Ω_i, namely

$$\psi_n = \Omega_0\,, \qquad \psi_j = (-1)^{n-j}\,\Omega_{n-j} + 1 \qquad (0 < j < n)\,.$$

The relations (14), (15) are due to E. A. Maxwell and J. A. Todd [9]. As special cases one gets the relations of Segre and Severi [9]. We state one more relation [9] which can be derived from (14). If n is odd, then

$$\sum_{j=0}^{n} 2^{n-j}\psi_j = 0 . \tag{16}$$

Remark. In the case that the ψ_j are integers, this formula implies

$$c_1^n \equiv 0 \ (\mathrm{mod}\, 2) .$$

In fact, we have for an orientable manifold M^{4k+2} with vanishing Stiefel-Whitney class w_3 that

$$0 = Sq^2\, w_2^{2k} = w_2^{2k}\, U^2 = w_2^{2k+1}$$

(see note [1] Theorem I′).

5. Steenrod's reduced powers in almost complex manifolds

Theorem I of the note [2] gives a formula for the classes s_q^r in terms of Pontrjagin classes. Theorem I′ gives a formula for the classes of Wu (Steenrod squares). By using formulas 1,(7) and Lemma 1.4 we can summarize these results for the special case of an almost complex manifold in the following theorem.

Theorem 5.1. *Let M_n be almost complex and let $c_i \in H^{2i}(M_n,\mathbb{Z})$ be the Chern classes of M_n. For $k+2r(q-1)=2n$ the class $s_q^r \in H^{2r(q-1)}(M_n,\mathbb{Z})$ is defined by*

$$\mathscr{P}_q^r v = s_q^r v \quad \textit{for all} \quad v \in H^k(M_n,\mathbb{Z}_q) .$$

We have the formula

$$s_q^r \equiv q^r\, T_{r(q-1)}(c_1, c_2, \ldots, c_{r(q-1)}) \bmod q .$$

The classes s_q^r are obtained from the Chern classes mod q *by the power series*

$$1 + (x - x^q + x^{q^2} - x^{q^3} + \ldots)^{q-1} .$$

For the classes U^i of Wu ($U^i \in H^i(M_n,\mathbb{Z}_2)$) defined by

$$Sq^i\, v = U^i v \quad \textit{for all} \quad v \in H^{2n-i}(M_n,\mathbb{Z}_2) ,$$

we have the formula

$$U^{2r+1} = 0 , \qquad U^{2r} \equiv 2^r\, T_r(c_1, \ldots, c_r) \bmod 2 .$$

The classes U^{2r} $(r=1,2,..,n)$ are obtained from the Chern classes mod 2 *by the power series $1+x+x^2+\ldots+x^{2^i}+\ldots$. We define $s_2^r = U^{2r}$ and have the formula*

$$s_q^r = q^r\, T_{r(q-1)}(c_1, \ldots, c_{r(q-1)}) \ \textit{for all primes.}$$

We are going to give some applications of Theorem 5.1.

Theorem 5.2. *Let M_n be almost complex and q a prime. If $r > \left[\frac{n}{q}\right]$, then*

$$q^r \, T_{r(q-1)}(c_1, c_2, \ldots, c_{r(q-1)}) \equiv 0 \pmod q .$$

Note. These congruences are not trivial (see the Lemma 1.5). Now we give a theorem of Borel-Serre [10] on unitary sphere bundles (in a slightly different formulation) and then the application of this theorem to the tangential bundle of an almost complex manifold. We write $\mathscr{P}_2^r$ instead of Sq^{2r}.

Let $[E, X, U(n)]$ be a unitary $(2n-1)$-sphere bundle with the Chern classes $c_i \in H^{2i}(X, \mathbb{Z})$. Write formally

$$1 + c_1 x + c_2 x^2 + \ldots + c_n x^n = (1 + u_1 x)(1 + u_2 x) \ldots (1 + u_n x),$$

i.e., x is an indeterminate, the c_i are the elementary symmetric functions in the u_i. In the following formulas only symmetric functions in the u_i occur which have to be regarded as polynomials in the c_i. A symmetric function will be characterized by its generating term. For example

$$\sum u_1^2 \cdot u_2^3 := \sum_{i_1, i_2} u_{i_1}^2 \cdot u_{i_2}^3 .$$

Theorem 5.3 (Borel-Serre [10]). *For the Chern classes $c_1, c_2, \ldots, c_n$ of $[E, X, U(n)]$ we have the following formulas* mod q *($q \geqq 2$):*

$$\mathscr{P}_q^r c_k = \mathscr{P}_q^r \sum u_1 u_2 \ldots u_k = \sum u_1^q u_2^q \ldots u_r^q u_{r+1} \ldots u_k \; (r \leqq k) .$$

In particular $(q \leqq n+1)$

$$\text{(17)} \quad \mathscr{P}_q^1 c_{n-q+1} = c_{n-q+1} \sum u_1^{q-1} - c_{n-q+2} \sum u_1^{q-2} + \ldots - c_{n-1} c_1 + n c_n .$$

Remark. In case $q = n+1$, $\mathscr{P}_q^1 c_{n-q+1} = 0$ and (17) is an algebraic identity, namely the familiar recursion formula for the sum $\sum u_1^r$.

5 **Corollary 5.3.** *If for an $[E, X, U(n)]$ the Chern classes $c_1, c_2, \ldots, c_{n-1}$ vanish, then $c_n \equiv 0$* mod q *for all primes $q \leqq n$ which do not divide n.*

In the case that $[E, X, U(n)]$ is the tangential bundle of an almost complex manifold M_n we can compare (17) with the formulas $(2 \leqq q \leqq n+1)$

$$\mathscr{P}_q^1 c_{n-q+1} = c_{n-q+1} s_q^1 = c_{n-q+1} \sum u_1^{q-1}$$

and obtain

Theorem 5.3*. *For the Chern classes $c_1, c_2, \ldots, c_n$ of M_n we have the following formulas* mod q $(2 \leqq q \leqq n+1)$:

$$c_{n-q+2} \sum u_1^{q-2} - c_{n-q+3} \sum u_1^{q-3} + \ldots + c_{n-1} c_1 - n c_n \equiv 0 \pmod q .$$

For example, $n c_n = 0 \pmod 2$, $c_{n-1} c_1 - n c_n \equiv 0 \pmod 3$,

$$c_{n-3}(3c_3 - 3c_1 c_2 + c_1^3) - c_{n-2}(c_1^2 - 2c_2) + c_{n-1} c_1 - n c_n \equiv 0 \pmod 5 .$$

If all the characteristic numbers $c_{i_1}\cdot c_{i_2}\cdot\ldots\cdot c_{i_r}$ with $n-1\geqq i_1\geqq i_2\geqq\ldots\geqq i_r$ and $i_1+i_2+\ldots+i_r=n$ vanish, then c_n (= the Euler-Poincaré characteristic of M_n) is divisible by all primes $q\leqq n+1$ which do not divide n.

Corollary 5.3*. *If in an almost complex manifold M_n the Chern classes $c_1, c_2, \ldots, c_{\left[\frac{n}{2}\right]}$ vanish, then the Euler-Poincaré characteristic of M_n is divisible by all primes $q\leqq n+1$ which do not divide n.*

Remark. Corollary 5.3* implies that the spheres S^{2k}, except S^2, S^6, are not almost complex (Borel-Serre, only S^4 escapes Corollary 5.3). It seems to be
6 interesting to ask for sharper theorems on the Euler-Poincaré characteristic of
an almost complex manifold M_n for which all characteristic numbers

$$c_{i_1}\cdot c_{i_2}\cdot\ldots\cdot c_{i_r}(n-1\geqq i_1\geqq i_2\geqq\ldots\geqq i_r,\, i_1+i_2+\ldots+i_r=n)$$

vanish. In the last section we shall return to this problem. For which complex dimensions n does there exist an almost complex manifold M_n for which these characteristic numbers vanish and for which $c_n\neq 0$? For which complex dimensions n does there exist an almost complex manifold with $c_1=c_2=\ldots=c_{\left[\frac{n}{2}\right]}=0$ and $c_n\neq 0$? (The algebraic surface of degree 4 in P_3 supplies an example to the second question for $n=2$. In this example $c_1=0$ and $c_2=24$.)

Remark. By combining Theorems 5.1 and 5.3 one can obtain numerous congruences for the Chern classes for an almost complex manifold M_n.

6. The Todd genus for manifolds of type (π_r)

In Sections 6 and 8 we study properties of the Todd genus of an almost complex manifold M_n. In particular, we prove in Section 8 that $2^{n-1}T(M_n)$ *is always an integer.* In Section 7 we make some general preparations for Section 8.

We say that an almost complex manifold M_n of n complex dimensions is of type (π_r), if there exist elements $u_1, u_2, \ldots, u_r\in H^2(M_n, \mathbf{Z})$ such that

$$1+c_1+c_2+\ldots+c_n=(1+u_1)(1+u_2)\ldots(1+u_r)$$

(c_i denotes the Chern classes of M_n, $c_i\in H^{2i}(M_n, \mathbf{Z})$). For example, the complex projective space P_n is of type (π_{n+1}). If M_n is of type (π_n), we say simply "M_n is of type (π)". For a manifold M_n of type (π_r) we can express the Todd genus by means of the function I.

Theorem 6.1. *Let M_n be of type (π_r) and*

$$1+c_1+c_2+\ldots+c_n=(1+u_1)(1+u_2)\ldots(1+u_r)\,.$$

We have

$$2^n T(M_n) = I(M_n) + \sum_{i=1}^{r} I(u_i) + \sum_{1 \leq i_1 < i_2 \leq r} I(u_{i_1}, u_{i_2}) + \dots$$
$$+ \sum_{1 \leq i_1 < \dots < i_r \leq r} I(u_{i_1}, u_{i_2}, \dots, u_{i_r}) ,$$
$$2^n T(M_n) = I(M_n) + \sum_{s=1}^{r} \sum_{1 \leq i_1 < \dots < i_s \leq r} I(u_{i_1}, u_{i_2}, \dots, u_{i_s}) .$$

Proof. ($\varkappa$ denotes the $2n$-dimensional component.) We have the following formulas:

$$\frac{-2x}{e^{-2x}-1} = \frac{x}{\operatorname{tgh} x} + x ,$$

$$2^n T(M_n) = \frac{-u_1}{e^{-u_1}-1} \cdot \frac{-u_2}{e^{-u_2}-1} \dots \frac{-u_r}{e^{-u_r}-1} = \prod_{i=1}^{r} \left(u_i + \frac{u_i}{\operatorname{tgh} u_i} \right) ,$$
$$\sum_{j=0}^{\infty} L_j(p_1(M_n), \dots) = \sum_{j=0}^{\infty} \bar{L}_{2j}(c_1(M_n), \dots) = \prod_{i=1}^{r} \frac{u_i}{\operatorname{tgh} u_i} .$$

(See Lemma 1.2.)

These formulas together with Theorem 3.2 establish the proof.

Theorem 6.1 yields the

Lemma 6.2. *The number $2^n T(M_n)$ is an integer for every almost complex manifold of type (π_r).*

7

7. Fiberings with the flag manifold $U(n)/T^n$ as fibre

We shall prove that $2^{n-1} T(M_n)$ is always an integer by constructing, for every M_n, an almost complex manifold (of higher dimension) which is of type (π) and the Todd genus of which coincides with $T(M_n)$. Before doing so we have to make some general preparations. (See Borel [12] and Chern [13].)

For a group G and a subgroup U we define G/U by "$g_1 \sim g_2 \leftrightarrow g_1^{-1} g_2 \in U$" ($g_1, g_2 \in G$). The group G regarded as group of *left* translations of G operates on G/U.

Let $U(n)$ be the unitary group regarded as group of all unitary $n \times n$ matrices. The group $U(n_1) \times U(n_2) \times \dots \times U(n_k)$ with $n_1 + n_2 + \dots + n_k = n$ is a subgroup of $U(n)$

$$\begin{pmatrix} \boxed{U(n_1)} & & & \\ & \boxed{U(n_2)} & & \\ & & \ddots & \\ & & & \boxed{U(n_k)} \end{pmatrix} \subset U(n) .$$

For $\underbrace{U(1) \times \dots \times U(1)}_{k \text{ times}} \times U(n-k)$ we write $T^k \times U(n-k)$. In particular, T^n is the subgroup of all diagonal matrices (maximal torus of $U(n)$). By

$T^{k-1}\times 1\times U(n-k)$ we denote the subgroup of $T^k\times U(n-k)$ consisting of those matrices which have 1 at the k^{th} place of the diagonal.

The manifold $U(n)/T^n$ is the so-called flag manifold (= space of all n frames of unitary orthogonal complex line elements in the origin of the n dimensional vector space over the complex numbers). The manifold $U(n)/T^n$ is almost complex ($U(n)/T^n$ is even a rational algebraic manifold). The manifold $U(n)/T^n$ can be fibred in complex projective lines P_1:

$$U(n)/T^n \xrightarrow{P_1} U(n)/T^{n-2}\times U(2)\,.$$

Moreover, we get a sequence of fiberings in complex projective spaces

$$\begin{aligned} U(n)/T^{n-2}\times U(2) &\xrightarrow{P_2} U(n)/T^{n-3}\times U(3)\\ U(n)/T^{n-3}\times U(3) &\xrightarrow{P_3} U(n)/T^{n-4}\times U(4)\\ &\ \ \vdots\\ U(n)/T^1\times U(n-1) &\xrightarrow{P_{n-1}} U(n)/U(n)\,. \end{aligned}$$

The bundle

$$U(n)/T^k\times U(n-k) \xrightarrow{P_{n-k}} U(n)/T^{k-1}\times U(n-k+1) \quad (1\leqq k\leqq n) \tag{18}$$

admits $U(n-k+1)$ as structure group with the associated bundle

$$U(n)/T^{k-1}\times 1\times U(n-k) \xrightarrow{S^{2(n-k)+1}} U(n)/T^{k-1}\times U(n-k+1)\,. \tag{19}$$

We also have the fiberings ($1\leqq k\leqq n$)

$$U(n)/T^{k-1}\times 1\times U(n-k) \xrightarrow{S^1} U(n)/T^k\times U(n-k)\,. \tag{20}$$

The manifolds $U(n)/T^k\times U(n-k)$, $0\leqq k\leqq n$, occurring in the fiberings (18) have a natural almost complex structure compatible with the fiberings. The group $U(n)$ operates on all manifolds occuring in (18), (19), (20) in such a way that these fiberings are preserved. Therefore, if $[E,X,U(n)]$ is a unitary $(2n-1)$-sphere bundle, we may construct the associated bundles $\mathscr{F}_k\to X$ and $\mathscr{S}_k\to X$ with $U(n)/T^k\times U(n-k)$ (resp. $U(n)/T^{k-1}\times 1\times U(n-k)$) as fibre ($1\leqq k\leqq n$), and we get a sequence of fiberings which we illustrate by the following commutative diagram.

$$\begin{tikzcd} \mathscr{F}_n \arrow[d] \arrow[dr] & \mathscr{F}_{n-1} \arrow[d] \arrow[dr] & \mathscr{F}_{n-2} \arrow[d] \arrow[dr] & \mathscr{F}_{n-3} \arrow[d] & & & \mathscr{F}_1=E \arrow[d] \arrow[dr] & \\ \mathscr{S}_n \arrow[r, "P_0"'] & \mathscr{S}_{n-1} \arrow[r, "P_1"'] & \mathscr{S}_{n-2} \arrow[r, "P_2"'] & \mathscr{S}_{n-3} \arrow[r] & \dots \arrow[r] & & \mathscr{S}_1 \arrow[r, "P_{n-1}"'] & X=\mathscr{S}_0 \end{tikzcd} \tag{21}$$

We denote by u_r the characteristic class of the S^1-bundle $\mathscr{F}_r\to\mathscr{S}_r$ ($u_r\in H^2(\mathscr{S}_r,\mathbf{Z})$). The cohomology ring $H^*(\mathscr{S}_r,\mathbf{Z})$ may be regarded as a subring of $H^*(\mathscr{S}_{r+1},\mathbf{Z})$. We observe that the bundle $\mathscr{F}_{r+1}\to\mathscr{S}_r$ lifted up to $\mathscr{S}_{r+1}$ splits up over $\mathscr{S}_{r+1}$ in the bundles $\mathscr{F}_{r+1}\to\mathscr{S}_{r+1}$ and $\mathscr{F}_{r+2}\to\mathscr{S}_{r+1}$. By successive applications of the Whitney duality theorem for Chern classes we obtain by induction over k ($k=n-r$, $k=1,2,\dots,n$).

Lemma 7.1. *The Chern class of the bundle*

$$\mathscr{F}_{n-k+1} \xrightarrow{S^{2k-1}} \mathscr{S}_{n-k} \ \textit{is}\ \ (1+u_{n-k+1})(1+u_{n-k+2})\dots(1+u_n)\,.$$

Now we can apply Theorem 3.1 (Remark 2) of the note [1] to the complex-projective bundle

$$\mathscr{S}_{n-k+1} \xrightarrow{P_{k-1}} \mathscr{S}_{n-k}\,.$$

That is, we consider the space $V(P_{k-1})$ of all unit tangential vectors of P_{k-1}

$$V(P_{k-1}) \xrightarrow{S^{2k-3}} P_{k-1}$$

and the associated bundle $\mathscr{S}_{n-k+1,V}$ with fibre $V(P_{k-1})$

(22)
$$\begin{tikzcd} & \mathscr{S}_{n-k+1,V} \arrow[r, "V(P_{k-1})"] \arrow[d, "S^{2k-3}"] & \mathscr{S}_{n-k} \\ \mathscr{F}_{n-k+1} \arrow[r, "S^1"] & \mathscr{S}_{n-k+1} \arrow[ur, "P_{k-1}"'] & \end{tikzcd}$$

As indicated in the diagram, $\mathscr{S}_{n-k+1,V}$ is a unitary $(2k-3)$-sphere bundle over $\mathscr{S}_{n-k+1}$. By Theorem 3.1 of the note [1] we infer that the *Chern class* of this bundle is given by

$$(23)\qquad c\,(\mathscr{S}_{n-k+1,V}, \mathscr{S}_{n-k+1}, U(k-1)) = \prod_{j=n-k+1}^{n} (1+u_j-u_{n-k+1})\,.$$

Repeated application of formula (23) leads to the

Lemma 7.2. *Let M_r be an almost complex manifold and denote by $1+c_1+c_2+\ldots+c_r$ the Chern class of its tangential bundle. Let*

$$E \xrightarrow{S^{2n-1}} M_r$$

be a unitary $(2n-1)$-sphere bundle over M_r with the Chern class $1+d_1+d_2+\ldots+d_n$. Make the construction of the diagram (21) with respect to the bundle E. Then we arrive at a manifold $\mathscr{S}_{n-1}$ of complex dimension

$$r+\frac{n(n-1)}{2}$$

which has a natural almost complex structure compatible with the fiberings of the diagram (21):

$$\mathscr{S}_{n-1} \xrightarrow{U(n)/T^n} M_r\,.$$

We have the formulas ($u_i \in H^2(\mathscr{S}_{n-1}, \mathbb{Z})$ and $c_i, d_i \in H^{2i}(M_r, \mathbb{Z})$):

$$1+d_1+d_2+\ldots+d_n = (1+u_1)(1+u_2)\ldots(1+u_n)\,.$$

The Chern class of the tangential bundle of $\mathscr{S}_{n-1}$ is

$$(1+c_1+c_2+\ldots+c_r)\cdot\prod_{i>j}(1+u_i-u_j)\,.$$

Remark. The lemma applies in particular to the case that M_r is a point. We obtain a formula for the Chern class of the tangential bundle of the manifold $U(n)/T^n$

$$(24)\qquad c\,(U(n)/T^n) = \prod_{i>j}(1+u_i-u_j)\,.$$

Hence $U(n)/T^n$ is of type (π). We have

$$u_i \in H^2(U(n)/T^n, \mathbb{Z})$$

and

$$(1+u_1)(1+u_2)\dots(1+u_n)=1\,. \tag{25}$$

The cohomology ring $H^*(U(n)/T^n,\mathbb{Z})$ is generated by the $u_i (1 \leqq i \leqq n)$ with the only relation (25)(Borel [12]). By the method employed in this section the Chern class of the manifold

$$U(n)/(U(n_1)\times U(n_2)\times\dots\times U(n_k))\,,\qquad n_1+n_2+\dots+n_k=n\,,$$

8 can be determined. Moreover, A. Borel pointed out to the author that (24) can be generalized to an arbitrary compact Lie group G: Let T be a maximal torus. The manifold G/T is a complex manifold of type (π). Namely, if $r_1,\dots,r_m \in H^2(G/T,\mathbb{Z})$ are a choice of "positive roots" of G, then G/T has an almost complex structure with $c(G/T)=\prod_{i=1}^{m}(1+r_i)$, m = complex dimension of G/T. In particular, the product of all roots is the Euler-Poincaré characteristic of G/T.

8. The Todd genus. Manifolds of type (γ)

We are now able to prove the

Theorem 8.1. *The number $2^{n-1}T(M_n)$ is an integer for every compact almost complex manifold.*

Proof. We construct the almost complex manifold $\mathscr{S}_{n-1}$ of Lemma 7.2 taking as bundle E the tangential bundle of M_n (hence $r=n$ and $c_i=d_i$). For the Chern class of $\mathscr{S}_{n-1}$ we have the formula

$$c(\mathscr{S}_{n-1})=(1+u_1)(1+u_2)\dots(1+u_n)\prod_{i>j}(1+u_i-u_j)\,.$$

Therefore, the manifold $\mathscr{S}_{n-1}$ is of type (π). We know from Theorem 4.4 of the note [1] that

$$T(M_n)=T(\mathscr{S}_{n-1})\,.$$

By Lemma 6.2 the Todd genus $T(M_n)$ multiplied with a power of 2 is an integer. The Todd polynomial T_n is a polynomial with (relatively prime) integer coefficients divided by a big denominator (Lemma 1.5). This denominator contains 2 exactly n times. Hence $2^n T(M_n)$ is an integer which by Theorem 5.2 is even. This completes the proof.

No example of an almost complex manifold M_n is known where $T(M_n)$ is not an integer. R. Thom has communicated to the author that $T(M_2)$ and $T(M_3)$ are always integers. He pointed out that this fact is related to a recent theorem of Rohlin [13].

If the second Stiefel-Whitney class of an orientable compact differentiable manifold vanishes, then the Pontrjagin class $p_1(V^4)$ is divisible by 48.

Before showing how "$T(M_3)$ is an integer" can be derived from this theorem of Rohlin we state the following trivial lemma:

Lemma 8.2. *If the statement "$T(M_n)$ is an integer for all complex manifolds of complex dimension n" is true for $n = n_0$, then it is true for all $n < n_0$.*

Proof. Assume $n < n_0$. We have

$$T(M_n) = T(M_n) \times T(P_{n_0-n}) = T(M_n \times P_{n_0-n}) = \text{integer}.$$

Theorem 8.3 (Thom). *The Todd genus is an integer for all almost complex manifolds M_2, M_3.*

Proof. By Lemma 8.2 it is sufficient to prove that $T(M_3)$ is an integer. Represent in M_3 the Chern class c_1 by a subvariety V^4 (not necessarily almost complex). By Whitney duality the second Stiefel-Whitney class of V^4 is 0. By 3,(9a) we find that

$$p_1(V^4) = -2\,c_1\,c_2.$$

By Rohlin $-2\,c_1\,c_2 \equiv 0\ (48)$. Q.E.D.

We are now returning to the question discussed in Theorem 5.3* (Remark).
9 For brevity we say that an almost complex manifold M_n is of type (γ) when all characteristic numbers $c_{i_1}\cdot c_{i_2}\ldots c_{i_r}$ with $n-1 \geqq i_1 \geqq i_2 \geqq \ldots \geqq i_r$ and $i_1 + i_2 + \ldots + i_r = n$ vanish. We say that M_n is of type (γ') when

$$c_1, c_2, \ldots, c_{\left[\frac{n}{2}\right]}$$

vanish. Obviously (γ') implies (γ). For a manifold M_{2k} of type (γ) *the Todd genus and the index are rational multiples of the Euler-Poincaré characteristic c_{2k} of M_{2k}*. By 1,(6) and 1,(8) we obtain the formulas (B_k: Bernoulli number):

(26) $$I(M_{2k}) = (-1)^k \frac{2^{2k+1}(2^{2k-1}-1)}{(2k)!} B_k \cdot c_{2k},$$

(27) $$T(M_{2k}) = (-1)^{k-1} \frac{B_k}{(2k)!} \cdot c_{2k},$$

(28) $$I(M_{2k}) = -2^{2k+1}(2^{2k-1}-1)\,T(M_{2k}).$$

Because $2^{2k-1}T(M_{2k})$ is an integer we obtain by (27) divisibility properties for c_{2k}. The question whether these divisibility properties formally imply those of Theorem 5.3* seems to be of a difficult arithmetic nature involving curious properties of the Bernoulli numbers B_k. Since B_k is always $\neq 0$, we can see that for a manifold M_{2k} of type (γ) either T, I, c_{2k} are all 0 or $\neq 0$. The formulas (26), (27), (28) are of a rather questionable character, because of missing examples (see the two questions, 5.3* Remark).

Footnotes

[1] F. Hirzebruch, Todd arithmetic genus for almost complex manifolds, Notes, Princeton University, 1953.
[2] F. Hirzebruch, On Steenrod reduced powers in oriented manifolds, Notes, Princeton University, 1953.
[3] R. Thom, C. R. Acad. Sci., Paris, *236*, 453, 573, 1733 (1953).
[4] V. A. Rohlin, Doklady Akad. Nauk. SSSR (N.S.) *84*, 221–224 (1952).
[5a] J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, New York, 1939.
[5b] N. Nielsen, Traité élémentaire des nombres de Bernoulli, Paris, 1923.
[6] J. Adem, Proc. Nat. Acad. Sci., U.S.A., *38*, 720–726 (1952).
[7] A special case of this formula in F. Hirzebruch, J. reine angew. Math. *191*, 110–124 (1953).
[8] The Euler characteristic $\chi(D)$ was introduced by Kodaira-Spencer (On arithmetic genera of algebraic varieties, Proc. Nat. Acad. Sci., U.S.A. (to appear)) and independently by J-P. Serre.
[9] E. A. Maxwell and J. A. Todd, Proc. Cambridge Philos. Soc., *33*, 438–443 (1937).
[10] A. Borel and J-P. Serre, C. R. Acad. Sci., Paris, *233*, 680–682 (1951).
[11] E. G. Kundert, Annals of Math., *54*, 215–246 (1951).
[12] A. Borel, Annals of Math., *57*, 115–207 (1953).
[13] S. S. Chern (to appear in Amer. J. Math.).

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