5.

On Steenrod reduced powers in oriented manifolds

Vervielfältigtes Manuskript, Princeton University, 1953

Introduction. We use the notation of a preceding note [1]. Let M^m be a compact *oriented* differentiable manifold. The reduced powers [2] are defined for every odd prime q:

$$\mathscr{P}_q^r\colon H^k(M^m, \mathbf{Z}_q) \to H^{k+2r(q-1)}(M^m, \mathbf{Z}_q).$$

In case $k+2r(q-1)=m$, there exists by Poincaré duality an element $s_q^r \in H^{2r(q-1)}(M^m, \mathbf{Z}_q)$ such that

$$\mathscr{P}_q^r u = s_q^r u \quad \text{for all} \quad u \in H^k(M^m, \mathbf{Z}_q).$$

We shall express s_q^r as a polynomial in the Pontrjagin classes $p_1, p_2, p_3, \dots$ of M^m where $p_i \in H^{4i}(M^m, \mathbf{Z})$. We shall use the formalism of the multiplicative sequences described in the note [1]. For formal reasons (since the Pontrjagin classes have "quaternion" dimensions) we replace the indeterminate x in the power series by z.

Let $\{L_j\}$ be the multiplicative sequence (rational coefficients) corresponding to the power series

$$Q(z) = \sqrt{z}\,(\operatorname{tgh}\sqrt{z})^{-1} = 1 + \frac{z}{3} - \frac{z^2}{45} + \frac{2z^3}{945} + \dots.$$

One has $3L_1 = a_1$, $45L_2 = 7a_2 - a_1^2$, $945L_3 = 62a_3 - 13a_2a_1 + 2a_1^3$.

Theorem I. *The class s_q^r can be expressed as a polynomial in the Pontrjagin classes:*

$$s_q^r = [q^r L_{1/2(q-1)r}(p_1, \dots, p_{1/2(q-1)r})]_q,$$

i.e., *$L_{1/2(q-1)r}$ multiplied by q^r is a polynomial with coefficients which are integers* mod q *(do not contain q in the denominators) and which reduced* mod q *is s_q^r.*

For example:

$$\begin{aligned}
&s_3^1 = p_1, \quad s_3^2 = -p_2 + p_1^2, \quad s_3^3 = p_3 + p_2p_1 + p_1^3, && \dots \quad (\text{mod } 3)\\
&s_5^1 = 3p_2 + p_1^2, && \dots \quad (\text{mod } 5)\\
&s_7^1 = 3p_3 - 3p_1p_2 + p_1^3, && \dots \quad (\text{mod } 7)
\end{aligned}$$

Since $s_q^r = 0$ for $r > \left[\dfrac{m}{2q}\right]$, we obtain from Theorem I many divisibility relations for the Pontrjagin classes of an M^m. Here we note only the following:

$$\begin{gathered}
M^4\colon\ p_1 \equiv 0\,(3), \qquad M^8\colon\ 7p_2 - p_1^2 \equiv 0\,(15),\\
M^{12}\colon\ 62p_3 - 13p_1p_2 + 2p_1^3 \equiv 0\,(21), \dots\\
M^{2r(q-1)}\colon\ q^r L_{1/2(q-1)r}(p_1, \dots) \equiv 0\,(\text{mod } q).
\end{gathered}$$

The following theorem, which is a consequence of a theorem of Thom [4], gives stronger divisibility properties than those obtainable by Theorem I.

Theorem II. *For an M^{4k} denote the index of the (symmetric) intersection matrix of the $2k$-dimensional rational homology classes by $I(M^{4k})$. (The index of a matrix is the number of + signs minus the number of − signs in the diagonal form.) Then*

$$I(M^{4k}) = L_k(p_1, \dots, p_k).$$

Hence $L_k(p_1, \dots, p_k)$ for an M^{4k} is always an integer.

For example,

$$M^8\colon\ 7p_2 - p_1^2 \equiv 0\ (45).$$

We assumed that the prime q is odd. In case $q=2$ we consider the Steenrod squares Sq^i and arrive at theorems of Wu [5]. For an arbitrary M^m (not necessarily oriented) Wu defined the class $U^i \in H^i(M^m, \mathbb{Z}_2)$ by $Sq^i v = U^i v$ for all $v \in H^{m-i}(M^m, \mathbb{Z}_2)$. Let $w_1, w_2, \dots, w_m$ be the Stiefel-Whitney classes of M^m where $w_i \in H^i(M_m, \mathbb{Z}_2)$. Then:

Theorem I′. *We have $U^i = [2^i T_i(w_1, \dots, w_i)]_2$ where T_i is the Todd polynomial* [1] *and where $2^i T_i$ has coefficients which are integers* mod 2.

The proofs of I and I′ are based on the "diagonal" method of Thom [6] and Wu [3], [5] and on the topological interpretation of the multiplicative sequences. We sketch the necessary lemmas and the proof for I, but we omit the
1 analogous lemmas and proof for I′. Full details will appear elsewhere.

Applications of I, I′, II, especially to almost complex manifolds, will be given in a following note.

1. Algebraic preliminaries

Lemma 1.1. *$Q(z) = \sqrt{z}\,(\operatorname{tgh}\sqrt{z})^{-1}$ is the only (rational) power series with $\varrho_n(Q^{2n+1}) = 1$ for $n \geqq 0$.* ($\varrho_k =$ the coefficient of z^k.)

Let $\{L_j\}$ be the multiplicative sequence belonging to $Q(z) = \sqrt{z}\,(\operatorname{tgh}\sqrt{z})^{-1}$. We consider $\sqrt{z}\,(\operatorname{tgh}\sqrt{z})^{-1}$ modulo an odd prime q. We put $z = x^2$ and note that $x(\operatorname{tgh} x)^{-1} + x = -2x(e^{-2x}-1)^{-1}$. It is therefore sufficient to consider the series $-x(e^{-x}-1)^{-1}$ modulo q where q may also be the prime 2. We write for the moment $B(x) = -x(e^{-x}-1)^{-1}$, $B(x) = 1 + \dfrac{x}{2} + \sum_{k=1}^{\infty} \dfrac{b_{2k}}{(2k)!} x^{2k}$, where b_{2k} are the Bernoulli numbers. By using the theorem of von Staudt [7] which states that $b_{2k} + \sum \dfrac{1}{p}$ is an integer (the sum is over all primes with $2k$ divisible by $p-1$) and by easy number theory, we prove that $B(q^{1/q-1}x)$ is a power series with coefficients which do not contain q in the denominator.

Lemma 1.2. $[B(q^{1/q-1}x)]_q = 1 + (x - x^q + x^{q^2} - x^{q^3} + \ldots)^{q-1}$, i.e., *reduced* mod q *all coefficients "vanish" which contain a positive power of q in the numerator.*

Remark. The reduced series is a series in x^{q-1}. Hence, for $q > 2$, we may define a series $S(z)$ by replacing x^2 by z in $1 + (x - x^q + x^{q^2} - \ldots)^{q-1}$.

Lemma 1.3. $q^r T_{r(q-1)}$ $(q \geqq 2)$ *and* $q^r L_{1/2(q-1)r}$ (q odd) *have integers* mod q *as coefficients.*

Remark. We can even prove that these polynomials do not vanish mod q.

2. Multiplicative functors for orthogonal sphere bundles

We denote an $(n-1)$-sphere bundle with the base space X and the structure group $SO(n)$ by $(E, X, SO(n))$. For such a bundle the Pontrjagin classes p_i are defined [8]; $p_i \in H^{4i}(X, \mathbf{Z})$, $0 \leqq i < \infty$, $p_0 = 1$, $p_i = 0$ for $2i > n$; in the notation of Wu [8] $p_i = P_0^{4i}$. The Pontrjagin classes themselves do not obey Whitney duality, but the p_i reduced to a coefficient field Γ of characteristic $\neq 2$ do satisfy duality. A proof for this apparently does not exist in the literature, but follows immediately from the fact that the Pontrjagin classes (mod Γ) can be regarded as elementary symmetric functions [9].

Definition [1]: A "functor" f which attaches to every $(E, X, SO(n))$ an element $f(E, X, SO(n)) \in G(X, \Gamma)$ is called multiplicative, if it commutes with mappings and if it obeys Whitney duality.

Theorem 2.1. *Let the field Γ be of characteristic $\neq 2$. Let p'_i be the image of p_i under the natural homomorphism $H^{4i}(X, \mathbf{Z}) \to H^{4i}(X, \Gamma)$. If $\{K_j\}$ is a multiplicative Γ-sequence* [1], *then obviously $f = \sum_{j=0}^{\infty} K_j(p'_1, \ldots, p'_j)$ is a multiplicative functor. Every multiplicative functor f can be uniquely obtained in this way, and therefore has automatically vanishing components in the dimensions $4i+2$. Let $(S^{2m-1}, P_{m-1}, SO(2))$ be the Hopf fibering and let $g \in H^2(P_{m-1})$ generate $H(P_{m-1})$. Put $g^2 = z$; then $Q(z) = f(S^{2m-1}, P_{m-1}, SO(2))$, $m \to \infty$, is the characteristic power series of f.*

3. Reduced powers

Let X be a finite complex. We consider the group [1] $G(X, \mathbf{Z}_q)$. If $d \in G(X, \mathbf{Z}_q)$, then the (finite) sum $\sum_{r=0}^{\infty} \mathscr{P}_q^r d$ is also an element of $G(X, \mathbf{Z}_q)$. The $\mathscr{P}_q^r$ satisfy.

Lemma 3.1 (Steenrod [2]). $\mathscr{P}_q^r(ab) = \sum_{i+j=r} \mathscr{P}_q^i(a)\, \mathscr{P}_q^j(b)$ *where* $a, b \in H(X, \mathbf{Z}_q)$.

Lemma 3.2. *We have a (multiplicative) isomorphism $\mathscr{P}_q$ of $G(X,\mathbb{Z}_q)$ onto itself defined by $\mathscr{P}_q d = \sum_{r=0}^{\infty} \mathscr{P}_q^r d$. In particular,*

$$\mathscr{P}_q(1+u) = 1 + u + u^q \quad (u \in H^2(X,\mathbb{Z}_q)),$$
$$\mathscr{P}_q^{-1}(1+u) = 1 + u - u^q + u^{q^2} - u^{q^3} + \dots .$$

Obviously $\mathscr{P}_p$ commutes with mappings.

Let $1+p_1'+p_2'+\dots$ be the Pontrjagin classes (modulo the odd prime q) of $(E,X,SO(n))$. We take the power series $Q(z)=1+z^{1/2(q-1)}$, denote the corresponding multiplicative $\mathbb{Z}_q$-sequence by $\{B_{q,j}\}$ and introduce the classes $b_{q,j}=B_{q,j}(p_1',\dots,p_j')$; $b_{q,j}$ is different from 0 only in "quaternion" dimensions j which are multiples of $1/2(q-1)$. The class $b_{3,j}$ is p_j reduced mod 3. The class $b_{q,1/2(q-1)i}$ is equal to the class $R^{2i(q-1)}(\operatorname{mod} q)$ of Wu [3]. Moreover, $\sum_{j=0}^{\infty} b_{q,j}$ and $\mathscr{P}_q^{-1}\left(\sum_{j=0}^{\infty} b_{q,j}\right)$ are multiplicative functors in the sense of Theorem 2.1. We can write uniquely $\mathscr{P}_p^{-1}\left(\sum_{j=0}^{\infty} b_{q,j}\right) = \sum_{r=0}^{\infty} \bar{s}_q^r$, where $\bar{s}_q^r \in H^{2r(q-1)}(X,\mathbb{Z}_q)$.

Lemma 3.3. $\sum_{r=0}^{\infty} \bar{s}_q^r$ *is the multiplicative functor belonging to the series* $S(z) = 1 + (x - x^q + x^{q^2} - x^{q^3} + \dots)^{q-1}$ *where* $x^2 = z$. (See Lemma 1.2.)

Proof by computing $\sum_{r=0}^{\infty} \bar{s}_q^r$ for the Hopf fibering.

By using Lemmas 1.2 and 1.3 we get

Theorem 3.4. $\bar{s}_q^r = q^r L_{1/2(q-1)r}(p_1',\dots)$.

In the case where $n=2k$ and $(E,X,SO(2k))$ admits $U(n)$ as structure group, one has Chern classes $1, c_1, c_2, \dots, c_i \in H^{2i}(X,\mathbb{Z})$, where [8] $1 - p_1 + p_2 - p_3 + \dots = \left(\sum_{i=0}^{\infty} c_i\right)\left(\sum_{j=0}^{\infty} (-1)^j c_j\right)$ and the formula

$$\bar{s}_q^r = q^r T_{r(q-1)}(c_1',\dots), \qquad c_i' = c_i (\operatorname{mod} q).$$

4. The Thom [6]-Wu [3], [5] construction for reduced powers

For a $(n-1)$-sphere bundle $(E,X,SO(n))$ with the projection $\pi\colon E \to X$ we construct the associated bundle A over X with the n dimensional unit ball as fibre. Here A can be considered as the mapping cylinder of π and E as subspace of A. The projection of A onto X may be denoted by $\bar{\pi}$. The following theorem is due to Thom [6] (Γ is a field).

Theorem 4.1. *There exists a natural isomorphism ϱ^* of $H^r(X,\Gamma)$ onto $H^{r+n}(A,E;\Gamma)$, where $\varrho^* 1 = U \in H^n(A,E;\Gamma)$ satisfies: $\varrho^* a = \bar{\pi}^* a \cup U$,*

$a \in H(X,\Gamma)$. In case X is an oriented manifold, U is the dual in $A-E$ to the homology class represented by X.

Lemma 3.1 and Theorem 2.1 imply

Theorem 4.2. *By attaching to every $(E,X,SO(n))$ the element $\varrho^{*-1}\left(\sum_{r=0}^{\infty}\mathscr{P}_q^r U\right)$ of $G(X,\mathbb{Z}_q)$, we obtain a multiplicative functor which (computed for the Hopf fibering) has the power series $Q(z)=1+z^{1/2(q-1)}$. Hence*

$$\varrho^{*-1}\sum_{r=0}^{\infty}\mathscr{P}_q^r U=\sum_{j=0}^{\infty} b_{q,j}\,.$$

Now let M be a compact oriented differentiable manifold of dimension m. We want to apply Theorem 4.2 to the tangential bundle of M. Consider the diagonal map $M \to M \times M$ which maps M one-to-one on the diagonal Δ. The normal bundle of Δ in $M \times M$ is isomorphic to the tangential bundle of M. The class U of Theorem 4.1 may be regarded as the dual of Δ in $M \times M$. We make the same calculations as Wu [5] did for $q=2$ and Steenrod squares and obtain for the classes s_q^r of M (see Introduction and 3):

Theorem 4.3. $\sum_{j=0}^{\infty} b_{q,j}=\mathscr{P}_q\left(\sum_{r=0}^{\infty} s_q^r\right)$. *Hence* $s_q^r=\bar{s}_q^r=q^r L_{1/2(q-1)r}(p_1',\ldots)$.

This is Theorem I of the Introduction.

Theorem 4.4 (Wu [3]). *The $b_{q,j} \,(\mathrm{mod}\, q)$ are topological invariants of M (independent of the differentiable structure). In particular, for $q=3$, the Pontrjagin classes are topological invariants* mod 3.

5. The index of a manifold M^{4k}

In this section we consider compact *oriented* differentiable manifolds which are not necessarily connected. Thom [4] has made the class of all differentiable manifolds into an algebra Ω by identifying M^m and $\tilde{M}^m$ if there exists a bounded N^{m+1} with $\partial N^{mn+1}=M^m-\tilde{M}^m$, and by defining addition of manifolds as the union and multiplication as the topological product, these operations being compatible with the identifications. Thom considers the algebra $\Omega \otimes \mathbb{Q}$ ($\mathbb{Q}$ denoting the rationals) and states the Theorem: *$\Omega \otimes \mathbb{Q}$ is generated by the complex-projective spaces P_{2k} of $2k$ complex-dimensions* ($k>0$).

By Thom [6] (Corrollaire V, 11) we know that $I(M^{4k})=0$ for a bounding manifold (variété-bord). It is easy to check that $I(M \times N)=I(M)\cdot I(N)$. Hence we have:

Theorem 5.1. *$I(M^{4k})$ defines an additive and multiplicative homomorphism of $\Omega \otimes \mathbb{Q}$ into the rationals.*

For an M^{4k} we define $L(M^{4k})$ by $L_k(p_1,\dots,p_k)$, for an oriented $M^m (m \not\equiv 0(4))$ we define $L(M^m)=0$. By the Whitney duality theorem (modulo rationals) for the classes L_i we have $L(M \times N) = L(M) \cdot L(N)$. Because an arbitrary term $\prod_{i=0}^{k} (p_i)^{v_i}$ (with $\sum v_i = k$) vanishes for a bounding manifold, we have

Theorem 5.2. $L(M^{4k})$ *defines an (additive and multiplicative) homomorphism of* $\Omega \otimes \mathbb{Q}$ *into the rationals.*

The Pontrjagin class $\sum_{i=0}^{\infty} p_i$ of the complex-projective space P_{2k} is $(1+g^2)^{2k+1}$ (g = generator $\in H^2(P_{2k}, \mathbb{Z})$). From Lemma 1.1 we get

Theorem 5.3. $L(P_{2k}) = 1$. *The multiplicative sequence* $\{L_j\}$ *with the series* $\sqrt{z}\,(\mathrm{tgh}\sqrt{z})^{-1}$ *can be characterized by* $L(P_{2k}) = 1$ *for all* $k \geqq 0$. *Moreover* $I(P_{2k}) = L(P_{2k})$.

Since L and I are both additive and multiplicative homomorphisms of $\Omega \otimes \mathbb{Q}$ into the rationals $\mathbb{Q}$ which coincide on the generators P_{2k} of $\Omega \otimes \mathbb{Q}$ we conclude that $I(M^{4k}) = L(M^{4k})$ for all compact oriented differentiable mani-
2 folds M^{4k}. This is Theorem II of the Introduction.

Footnotes

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Princeton, New Jersey, June 24, 1953