4.

Todd arithmetic genus for almost complex manifolds

Vervielfältigtes Manuskript, Princeton University, 1953

* In a recent paper [1] Kodaira and Spencer discussed the two different definitions p_a, P_a of arithmetic genus of a non-singular algebraic variety V_n. They proved $p_a(V_n) = P_a(V_n)$. Kodaira [2] proved the conjecture of Severi that $P_a(V_n) = g_n(V_n) - g_{n-1}(V_n) + g_{n-2}(V_n) - \ldots + (-1)^{n-1} g_1(V_n)$, where $g_i(V_n)$ is the dimension of the complex vector space of the holomorphic differentials of degree i attached to V_n. If one defines $\Pi(V_n) = \sum_{i=0}^{n} (-1)^i g_i(V_n)$, one can express the results of Kodaira-Spencer in the following formula ($g_0(V_n) = 1$):

$$\Pi(V_n) = 1 + (-1)^n p_a(V_n) = 1 + (-1)^n P_a(V_n),$$

and one may say that three of the different definitions of arithmetic genus, namely Π, p_a, P_a, are now known to be equivalent.

On the other hand, Todd [3] in 1937 has given expressions for the arithmetic genus in terms of his canonical classes [4]. Hodge [5] proved for a wide class of algebraic varieties that the canonical classes (up to sign) coincide with the Chern classes. Denote by c_i the Chern class of complex dimension i of the non-singular algebraic variety V_n, i.e., $c_i \in H^{2i}(V_n, \mathbb{Z})$ = the $2i$-dimensional cohomology group of V_n with integer coefficients. Then we may define a fourth arithmetic genus of V_n, which we denote by $T(V_n)$, in the following way: Take the Todd expressions for the arithmetic genus and replace the canonical classes by the Chern classes $2T(V_1) = c_1$, $12T(V_2) = (c_1^2 + c_2)$, $24T(V_3) = c_1 \cdot c_2$, $720T(M_4) = (-c_4 + c_3 c_1 + 3c_2^2 + 4c_2 c_1^2 - c_1^4) \ldots$ In general $T(V_n)$ is a well-defined polynomial of complex dimension n in the Chern classes with rational coefficients. Hence it represents a multiple of the fundamental cocycle of V_n and may therefore be considered as a rational number.

Todd has proved that $1 + (-1)^n P_a(V_n) = T(V_n)$, where $T(V_n)$ is expressed in terms of the canonical classes! But because of some difficulty in the definition of the canonical classes themselves, and because it is not proved completely whether the Chern classes coincide with the canonical classes, the question whether $\Pi(V_n) = T(V_n)$ for all algebraic varieties still seems to be open. Kodaira [6] has proved that $\Pi(V_n) = T(V_n)$ for all algebraic varieties, which are a complete non-singular intersection of hypersurfaces in some projective space.

Since the Chern classes are defined for an arbitrary almost complex manifold, the Todd genus $T(M_n)$ can also be defined for such a manifold. In this note we give a new definition for the Todd polynomials which simplifies the calculations heretofore made. We prove for arbitrary almost complex manifolds theorems for $T(M_n)$ which are partially known for algebraic manifolds.

For example: if an almost complex manifold M_n is fibred in k-dimensional complex projective spaces P_k with an almost complex manifold B_{n-k} as base, the fibering being compatible with the almost complex structure of M_n, P_k, B_{n-k}, then $T(M_n) = T(B_{n-k})$.

It is probably true for an algebraic variety that $\Pi(V_n) = T(V_n)$. This would imply that $T(V_n)$ is an integer. This question will be the main object of a forthcoming note: *Is $T(M_n)$ an integer for an arbitrary almost complex manifold M_n?* (for $n=1$ this is trivially true since c_n is the Euler characteristic). We cannot answer this question, but by using Steenrod's reduced power operations [7] it is possible to obtain partial results, e.g., for an arbitrary almost complex M_2: $c_1^2 + c_2 \equiv 0\,(6)$, for an M_3: $c_1 \cdot c_2 \equiv 0\,(6)$, for an M_4: $-c_4 + c_3 c_1 + 3c_2^2 + 4c_2 c_1^2 - c_1^4 \equiv 0\,(30)$, etc. It turns out that there is a very close relation between the Steenrod reduced powers and the Todd polynomials. By using a recent theorem of Thom [8] we obtain that $2^{n-1}\,T(M_n)$ is an integer for every compact almost complex manifold.

1. Algebraic preliminaries

Let $\sum_{i=0}^{\infty} a_i x^i$, $a_0 = 1$, be the power series with the indeterminates a_i as coefficients, and let Γ be a commutative ring with unit.

Let $\{K_j\}$ be a sequence of polynomials, K_j being of weight j in the a_i and having coefficients in Γ $(K_0 = 1)$. If $C = \sum_{i=0}^{\infty} c_i x^i$ is an arbitrary power series, we denote by $K(C)$ the series $\sum_{j=0}^{\infty} K_j(c_1, \ldots, c_j)\, x^j$. We call $\{K_j\}$ a multiplicative Γ-sequence provided that K is a homomorphism, i.e., $K(AB) = K(A) \cdot K(B)$, where A and B are the power series $\sum_{i=0}^{\infty} a_i x^i$ and $\sum_{i=0}^{\infty} b_i x^i$ with indeterminates a_i, b_i as coefficients $(a_0 = b_0 = 1)$.

Lemma 1.1. *By attaching to a multiplicative Γ-sequence the power series $K(1+x) = 1 + \gamma_1 x + \gamma_2 x^2 + \ldots$ $(\gamma_i \in \Gamma)$, one obtains a one-to-one correspondence between the multiplicative Γ-sequences and the power series with coefficients in Γ, the constant term being 1.*

Proof. We construct for every $Q(x) = \sum_{i=0}^{\infty} \gamma_i x^i$ a unique multiplicative Γ-sequence. Writing formally

$$1 + a_1 x + \ldots + a_m x^m = (1 + \alpha_1 x)\,(1 + \alpha_2 x) \ldots (1 + \alpha_m x)\,,$$

we express $Q(\alpha_1 x)\, Q(\alpha_2 x) \ldots Q(\alpha_m x)$ as a power series with coefficients which are polynomials in the a_i:

$$Q(\alpha_1 x)\, Q(\alpha_2 x) \ldots Q(\alpha_m x) = \sum_{j=0}^{\infty} K_{j,m}(a_1, \ldots, a_j)\, x^j\,.$$

One verifies easily that $K_{j,m}$ does not depend on m for $j \leqq m$. We write $K_{j,j} = K_j$ and obtain the multiplicative Γ-sequence with $K(1+x) = Q(x)$. We denote the coefficient of x^k in a power series $Q(x)$ by $\varrho_k(Q)$. Now let Γ be the rationals.

Lemma 1.2. *$Q(x) = -x(e^{-x}-1)^{-1}$ is the only power series with $\varrho_n(Q^{n+1}) = 1$ for $n \geqq 0$.*

Lemma 1.3. *$Q(x) = -x(e^{-x}-1)^{-1}$ is the only power series with $\varrho_0(Q) = 1$, $\varrho_1(Q) = \frac{1}{2}$, $\varrho_n(Q(x)^n Q(-x)) = 0$ for $n \geqq 1$.*

Definition. The multiplicative sequence belonging to $Q(x) = -x(e^{-x}-1)^{-1}$ is called the Todd sequence. We denote the polynomials of the sequence by T_j and the corresponding homomorphism by T:

$$2T_1 = a_1, \quad 12T_2 = a_1^2 + a_2, \quad 24T_3 = a_1 a_2,$$
$$720T_4 = (-a_4 + a_3 a_1 + 3a_2^2 + 4a_2 a_1^2 - a_1^4).$$

The Todd polynomials have many formal properties [9]. Here we note only that T_{2i+1} is always divisible by a_1.

1 **Lemma 1.4.** *Let $d_1, d_2, \ldots, d_m$, η be indeterminates, $d_0 = 1$. If one reduces $T\left(\sum_{i=0}^{n}(1-\eta x)^{n-i} d_i x^i\right)$ modulo the ideal generated by $\left[\sum_{i=0}^{n}(-\eta)^{n-i} d_i\right]$, one has of course at most the power η^{n-1} in the T_j. Moreover, T_j does not contain η^{n-1} for $j \neq n-1$, and T_{n-1} contains $(-\eta)^{n-1}$ as summand.*

2. Multiplicative functors for unitary sphere bundles

Let $[E,X,n]$ denote a $(2n-1)$-sphere bundle E over the finite complex X as base space with the unitary group $U(n)$ as structure group. For every $[E,X,n]$ the Chern classes c_i are defined ($0 \leqq i < \infty$, $c_i \in H^{2i}(X,\mathbf{Z})$, $c_0 = 1$ and $c_i = 0$ for $i > n$). We put $c(E,X,n) = \sum_{i=0}^{\infty} c_i$ and call it the Chern class of $[E,X,n]$. The Chern classes have the following properties:

1) *Invariance under mappings:* if g maps X' into X, then $c(E',X',n) = g^* c(E,X,n)$. Here E' denotes the bundle induced by g.

2) *Whitney Duality Theorem* [10]:

$$c(E,X,n_1+n_2) = c(E_1,X,n_1) \cdot c(E_2,X,n_2).$$

Here E denotes the product bundle of the two bundles E_1, E_2 over the same base space X.

Definition. Let Γ be a commutative ring with unit. Denote by $G(X,\Gamma)$ the abelian group (cup-product) of all elements of the cohomology ring $H(X,\Gamma)$ which are of the form $d = \sum_{i=0}^{\infty} d_i$, $d_i \in H^{2i}(X,\Gamma)$; $d_0 = 1$. A "functor" f which attaches to every $[E,X,n]$ an element $f(E,X,n) \in G(X,\Gamma)$ is called multiplicative if it obeys 1) and 2).

By using the technique of Borel-Serre [11] and Chern [10] in which the Chern classes $1, c_1, c_2, \ldots, c_n$ may be regarded as elementary symmetric functions of two dimensional classes, one obtains:

Theorem 2.1. *Let c_i' be the images of the Chern classes under the natural homomorphism $H^{2i}(X,\mathbb{Z}) \to H^{2i}(X,\Gamma)$. If K_j is a multiplicative Γ-sequence, then obviously $f = \sum_{j=0}^{\infty} K_j(c_1', \ldots, c_j')$ is a multiplicative functor.*

Every multiplicative functor f can be uniquely obtained in this way. The series $Q(x)$ is determined by $f(S^{2m-1}, P_{m-1}, 1)$ for large m.

Note. Writing $f = \sum_{i=0}^{\infty} f_i$, $(f_i \in H^{2i})$, then generally $f_i \neq 0$ for $i > n$.

In the following examples, we represent the multiplicative sequence by its characteristic power series $Q(x)$.

$Q(x) = 1 + x \quad \to$ Chern classes c_i

$Q(x) = (1+x)^{-1} \to$ the $\bar{c}_i$-classes considered by Chern [10]

$Q(x) = 1 + x^2 \quad \to$ the Pontrjagin classes

$Q(x) = \dfrac{-x}{e^{-x}-1} \to$ the Todd classes

There will be other interesting examples which give the relations to the Steenrod power operations as mentioned in the introduction. These will be discussed in a forthcoming note.

Remark. Theorems analogous to 2.1 hold for orthogonal sphere bundles.

3. A Theorem on complex-projective bundles

Let $[L, X, P_{n-1}, p]$ be a projective bundle, i.e., L is a bundle over the finite complex X with the complex-projective space P_{n-1} as fibre, the projection p, and the group G of all projective transformations of P_{n-1} as structure group. We denote the space of all tangential unit vectors of P_{n-1} by $V(P_{n-1})$. This is a unitary $(2n-3)$-sphere bundle over P_{n-1}. The group G operates in a canonical fashion on $V(P_{n-1})$. Hence we can construct the associated bundle with $V(P_{n-1})$ as fibre, the total space of which we call L_V. This is a $(2n-3)$-sphere bundle over L: $[L_V, L, n-1]$. In case X is a manifold, L_V is the space of all unit vectors in L tangential to the fibres. We give without proof a formula for the rational Chern class of $[L_V, L, n-1]$.

It is known that p^* maps $H(X,R)$ isomorphically into $H(L,R)$ (R denoting the rationals). We consider $H(X,R)$ as a subring of $H(L,R)$. We can find an element $\eta \in H^2(L,R)$ which, restricted to the fibre P_{n-1}, is dual to the negative hyperplane P_{n-2} in P_{n-1} (generator of the cohomology of P_{n-1}).

There is a unique relation

$$\eta^n - d_1\,\eta^{n-1} + d_2\,\eta^{n-2} - \ldots + (-1)^n d_n = 0 \quad \text{with} \quad d_i \in H^{2i}(X,R)\,,$$

and $H(L,R)$ is generated by $H(X,R)$ and η together with this relation (Formula of Hirsch [10]).

Theorem 3.1. *We have* $c'(L_V, L, n-1) = \sum_{i=0}^{n} (1-\eta)^{n-i} d_i$, *where* c' *is the image of* c *under the canonical isomorphism* $H(L,\mathbb{Z}) \to H(L,R)$.

Remark 1. One verifies directly that this expression does not depend on the choice of η and that $c_n' = 0$.

Remark 2. In case $[L, X, P_{n-1}, p]$ admits $GL(n,\mathbb{C})$ and with this also $U(n)$ as structure group, η can be chosen integral, η is the characteristic class of a 1-sphere bundle E over L. The d_i are in this case the Chern classes of the associated $(2n-1)$-sphere bundle $[E, X, n]$. Restricted to P_{n-1}, η is the characteristic class of the Hopf fibering $S^{2n-1}/S^1 = P_{n-1}$ which is dual to the negative hyperplane P_{n-2} in P_{n-1}. For integer classes Theorem 3.1 reads

$$c(L_V, L, n-1) = \sum_{i=0}^{n} (1-\eta)^{n-i} d_i\,.$$

4. The Todd genus

Let f be a multiplicative functor in the sense of Section 2. For a compact almost complex manifold M_n we consider the tangential $(2n-1)$-sphere bundle $[V(M_n), M_n, n]$ (Chern class $c(M_n) = 1 + c_1 + c_2 + \ldots + c_n$). We call $f[V(M_n), M_n, n]$ the f-class of M_n and $f_n[V(M_n), M_n, n]$ the f-genus of M_n, which is to be considered as element of the coefficient domain Γ (multiple of the fundamental cocycle (M_n) of M_n in its natural orientation).

The Chern class of the complex projective space P_n is $(1+g)^{n+1}$ where g is dual to the positive hyperplane P_{n-1}. Hence from Lemma 1.2 we have:

Theorem 4.1. *The unique (rational) f-genus with value 1 on P_n for all n is given by the Todd sequence. We call this genus $T(M_n)$; it is a rational number. Hence for* P_n: $T(P_n) = \Pi(P_n) = 1$.

Let M_n be a complex manifold, construct the complex manifold $\sigma_p M_n$, i.e., puncture M_n in p and replace p by the P_{n-1} of all complex line elements in p (Hopf σ-process [12] or elementary birational transformation with p as fundamental point). Then $\sigma_p M_n$ is a "topological sum" of the two manifolds M_n, P_n. In the sum, M_n has its natural and P_n its "unnatural" orientation. Here $H(\sigma_p M_n)$ is generated by $H(M_n)$ and by one element g (dual to P_{n-1} in $\sigma_p M_n$) together with the relation $g^n = (-1)^{n-1}$ (fundamental cocycle).

Theorem 4.2 [13]. $c(\sigma_p M_n) = c(M_n) + (1-g)^n(1+g) - 1$. *Hence, because of Lemma 1.3, the Todd genus is the unique f-genus which has value 1 on P_1 and is invariant under the σ-process.*

Theorem 4.3. *Because the Todd classes obey the duality theorem, one has obviously for almost complex manifolds* $T(M \times N) = T(M) \cdot T(N)$ (Todd [3] proved this for dimension of $M \times N \leqq 6$).

Let L be a projective bundle (fibre P_{n-1}) over the almost complex manifold M_m as base. The tangential bundle of the manifold L "splits up" into two factors: the bundle of tangential vectors of L tangential to the fibres and the bundle induced by the projection from the tangential bundle of M_m. Hence L admits $U(m) \times U(n-1)$ as structure group and is therefore almost complex.

Lemma 1.4, Theorems 3.1, 4.1, and the duality theorem for Todd classes imply

Theorem 4.4. $T(L) = T(M_n)\, T(P_{n-1}) = T(M_m)$.

The Theorems 4.3, 4.4, and other properties of $T(M)$ not here discussed show that $T(M)$ behaves like the Euler characteristic. This is not too surprising, because one conjectures for algebraic manifolds $\Pi(M) = T(M)$, and $\Pi(M)$ is in fact an Euler characteristic [1].

Footnotes

[1] K. Kodaira and D. C. Spencer, "On arithmetic genera of algebraic varieties", Proc. Nat. Acad. Sci. (to appear).
[2] K. Kodaira, "The theory of harmonic integrals and their application to algebraic geometry," Notes, Princeton University, 1953.
[3] J. A. Todd, Proc. London Math. Soc., II, Ser., *43*, 190–225 (1937).
[4] J. A. Todd, Proc. London Math. Soc., II, Ser., *43*, 127–141 (1937), and *45*, 410–424 (1939).
[5] W. V. D. Hodge, Proc. London Math. Soc., III, Ser., *1*, 138–151 (1951).
[6] K. Kodaira, loc. cit. in footnote 2, p. 43.
[7] N. E. Steenrod, Proc. Nat. Acad. Sci., *39*, 213–223 (1953).
[8] R. Thom, C. R. Acad. Sci., Paris, *236*, 1733–1735 (1953).
[9] E. A. Maxwell and J. A. Todd, Proc. Cambridge Philos. Soc., *33*, 438–443 (1937).
[10] S. S. Chern (to appear).
[11] A. Borel and J-P. Serre, C. R. Acad. Sci., Paris, *233*, 680–682 (1951); A. Borel, Ann. of Math., *57*, 115–207 (1953).
[12] H. Hopf, Rend. Mat. e Appl., V, Ser., *10*, 169–182 (1951).
[13] J. A. Todd, Proc. Edinburgh Math. Soc., II, Ser., *5*, 117–124 (1937).

Princeton, New Jersey, June 24, 1953.