Topology Vol. 1, pp. 25—45. Pergamon Press, 1961. Printed in Great Britain

ANALYTIC CYCLES ON COMPLEX MANIFOLDS†

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(*Received* 1 *August* 1961)

INTRODUCTION

LET X be a complex manifold, Y a closed irreducible k-dimensional complex analytic subspace of X. Then Y defines or "carries" a $2k$-dimensional integral homology class y of X, although the precise definition of y presents technical difficulties.‡ A finite formal linear combination $\sum n_i Y_i$ with n_i integers and Y_i as above is called a complex analytic cycle, and the corresponding homology class $\sum n_i y_i$ is called a complex analytic homology class. If an integral cohomology class u corresponds under Poincaré duality to a complex analytic homology class we shall say that u is a *complex analytic cohomology class*. The purpose of this paper is to show that a complex analytic cohomology class u satisfies certain *topological conditions, independent of the complex structure of* X. These conditions are that certain cohomology operations should vanish on u, for example $Sq^3 u = 0$: they are all torsion conditions. We also produce examples to show that these conditions are not vacuous even in the restricted classes of (a) Stein manifolds and (b) projective algebraic manifolds.

We should emphasize that the subspaces Y_i above are allowed arbitrary singularities. If one insists that all the Y_i are *sub-manifolds* then much stronger conditions must be satisfied by u. For example, according to Thom [21], $Sq^{2k+1}u = 0$ for all k (with similar results for other primes). In Thom's work the complex structure does not really enter, only the *almost* complex structure of X is used. Our proofs however rely essentially on deep results in the theory of coherent analytic sheaves.

If X is a compact Kähler manifold then there is a well-known necessary condition on a cohomology class $u \in H^{2q}(X; \mathbf{Z})$ in order that it should be complex analytic. This is that the harmonic form defined by u should be of type (q, q). Hodge has conjectured [13] that, if X is a projective algebraic manifold, these conditions are also sufficient. For $q = 1$ this conjecture is true [14]. However, for $q \geqslant 2$ the results of this paper show that the conjecture, *in this strong form*, is false. It remains a possibility that the conjecture is true when reformulated in terms of *rational* cohomology.

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† This was presented by the first author at the International Colloquium on Differential Geometry and Topology, Zurich 1960.

‡ These difficulties vary according to the homology theory used. If one is prepared to assume that the pair (X, Y) can be triangulated then simplicial theory can be used. Borel and Haefliger have solved the problem by use of Čech theory. In this paper we shall adopt a third approach based essentially on singular theory.

If X is a Stein manifold then it is known [9] that every class $u \in H^2(H; \mathbf{Z})$ is complex analytic. Our results show that this theorem does not generalize to higher dimensions.

The cohomology operations mentioned above are the differentials d_r of the spectral sequence $H^*(X; \mathbf{Z}) \Rightarrow K^*(X)$ introduced in [4]. The conditions $d_r u = 0$ (all r) are equivalent to saying that u "corresponds" to an element $\xi \in K^*(X)$ in this spectral sequence. Suppose now that u is complex analytic and is represented by the subspace Y. Then our method consists in constructing a vector bundle ξ on X, using a resolution of the sheaf of germs of holomorphic functions on Y, and showing that this "corresponds" to u in the spectral sequence.

The contents of the various sections are as follows. In §1, in order to include the non-compact case, we extend some of the definitions of [4] from finite complexes to general spaces. We introduce an "inverse limit" $\mathscr{K}$ and an "inverse limit cohomology" $\mathscr{H}$. Besides being essential† in the non-compact case, $\mathscr{H}$ is a convenient technical tool which is used to advantage in §5. In §2 we summarize the basic facts concerning coherent analytic sheaves and we make a few applications. In §3 we introduce a construction for $K^*(X, Y)$ analogous to the difference cochain in cohomology. Applying the construction in §4 to the resolutions of sheaves given in §2 we define the "Grothendieck element" $\gamma(S)$. The essential point is that Grothendieck's construction‡ (cf. [6]), which associates to a subvariety Y of X an element of $K^0(X)$, can be *localised* in a neighbourhood of Y. The main theorem (6.1) follows easily from this if we assume the triangulation of complex spaces. However we avoid this assumption by proving a few technical results in §5. Then in §6 we prove the main theorem and construct the examples (6.3) and (6.5). In §6 we use a result on the operators d_{2p-1} of the spectral sequence. This result is proved in §7.

The material of this paper will be employed in a future publication to prove the Riemann–Roch theorem for analytic embeddings. Because of this we have given some of our results in greater generality than is necessary for proving the theorems of this paper.

The authors are indebted to J. F. Adams for many helpful suggestions.

§1. K FOR GENERAL SPACES

In [4] we defined, for any pair§ (X, Y) of finite CW-complexes, a ring $K^*(X, Y)$. We shall now extend this to any pair of spaces as follows.

DEFINITION (1.1). *Let X be a topological space, Y a subspace, then an element $\xi \in \mathscr{K}^*(X, Y)$ is a functor which assigns to any map $f : (A, B) \to (X, Y)$ of a pair (A, B) of finite simplicial complexes an element $f^! \xi \in K^*(A, B)$ such that*

(i) *$f^! \xi$ depends only on the homotopy class of f,*

(ii) *if*

$$\begin{array}{ccc} (A, B) & \xrightarrow{f} & (X, Y) \\ \uparrow g & \nearrow f_1 & \\ (A_1, B_1) & & \end{array}$$

is a commutative diagram, then

$$g^!(f^! \xi) = f_1^! \xi.$$

† See note added in proof on p. 45.

‡ The fact that Grothendieck's construction extended to the analytic case was pointed out by Grauert.

§ By a pair (X, Y) of complexes we shall always mean that Y is a subcomplex of X.

If $Y = \varnothing$, $X = B_G$ is the classifying space of a group G, then this definition coincides with the one given in [4; §4.6] for $\mathscr{K}^*(B_G)$. If (X, Y) is a pair of finite CW-complexes then it follows at once from the definition that $\mathscr{K}^*(X, Y) \cong K^*(X, Y)$ in a canonical manner. If (X, Y) is a pair of CW-complexes then

$$\mathscr{K}^*(X, Y) \cong \varprojlim_{\alpha} K^*(X_\alpha, Y_\alpha)$$

where X_α runs through all the finite sub-complexes of X and $Y_\alpha = Y \cap X_\alpha$. Our notation therefore agrees with that adopted in [3].

In a quite analogous fashion to (1.1) we can introduce cohomology groups $\mathscr{H}^q(X, Y)$. For a pair of CW-complexes (X, Y) we have

$$\mathscr{H}^q(X, Y) \cong \varprojlim_{\alpha} H^q(X_\alpha, Y_\alpha)$$

with the same notation as above. It is then easy to show that, if X is a countable complex, the natural map

$$\alpha : H^q(X, Y) \to \mathscr{H}^q(X, Y),$$

where H^q denotes singular cohomology, is an epimorphism (for a further discussion see [16; Lemma 2]). Since H^q and $\mathscr{H}^q$ are both invariants of singular homotopy type it follows that α is also an epimorphism if X, Y belong to the class $\mathscr{W}_0$ of [15] (spaces having the homotopy type of a countable CW-complex). In particular this applies when X, Y are manifolds with countable topology [15; cor. 1].

The elementary properties of K^* developed in [4] extend at once to $\mathscr{K}^*$. Thus $\mathscr{K}^*$ is a contravariant functor of homotopy type and $\mathscr{K}^*(X, Y)$ is a module over the ring $\mathscr{K}^*(X)$. Moreover we have a ring homomorphism

$$\mathrm{ch} : \mathscr{K}^*(X) \to \mathscr{H}^{**}(X; \mathbf{Q}),$$

where $\mathscr{H}^{**}$ is the direct *product* of the $\mathscr{H}^q$. On the other hand the exact sequence for a pair (X, Y) may no longer hold, and the same applies *a fortiori* to the spectral sequence $H^*(X; \mathbf{Z}) \Rightarrow K^*(X)$. However the operators d_r of this spectral sequence, regarded as higher order cohomology operations, can be defined on $\mathscr{H}^k(X; \mathbf{Z})$. For our purposes it is sufficient to define the statement "$\mathrm{d}_r u = 0$ for all r, $u \in \mathscr{H}^k(X; \mathbf{Z})$" to mean: for any map $f: A \to X$ of a finite simplicial complex A, we have $\mathrm{d}_r(f^*u) = 0$ for all r.

The following criterion for the vanishing of the d_r is implicit in [4, §2].

LEMMA (1.2). *Let A be a finite CW-complex, A^q its q-skeleton and let $v \in H^k(A; \mathbf{Z})$. Then $\mathrm{d}_s v = 0$ for all $s < r$ if and only if there exists*

$$u \in H^k(A^{k+r-1}, A^{k-1}; \mathbf{Z})$$

and $$\xi \in K^*(A^{k+r-1}, A^{k-1}),$$

such that $$\sigma(u) = v$$

and $$\mathrm{ch}\, \xi = \rho_*(u) + \text{higher terms},$$

where σ is the natural homomorphism

$$H^k(A^{k+r-1}, A^{k-1}; \mathbf{Z}) \cong H^k(A, A^{k-1}; \mathbf{Z}) \to H^k(A; \mathbf{Z})$$

and ρ_* is induced by the coefficient homomorphism

$$\rho : \mathbf{Z} \to \mathbf{Q}.$$

This leads to the following result for general spaces.

LEMMA (1.3). *Let $Y \subset X$ and suppose that any map $f : A \to X$ of a finite simplicial complex A of dimension $k-1$ is homotopic to a map g with $g(A) \subset Y$. Let $u \in \mathscr{H}^k(X, Y; \mathbf{Z})$, and let v be the image of u in $\mathscr{H}^k(X; \mathbf{Z})$. Suppose that there exists $\xi \in \mathscr{K}^*(X, Y)$ such that*

$$\text{ch } \xi = \rho_*(u) + \text{higher terms.}$$

Then $\mathrm{d}_r v = 0$ *for all r.*

Proof. Let $f : A \to X$ be any map of a finite simplicial complex A. By hypothesis (and the homotopy extension property) f is homotopic to a map $g : (A, A^{k-1}) \to (X, Y)$. Then

$$\text{ch } g^!\xi = \rho_*(g^*u) + \text{higher terms,}$$

and the image of g^*u in $H^k(A; \mathbf{Z})$ is f^*v. Hence, by (1.2), $\mathrm{d}_r(f^*v) = 0$ for all r. Since this holds for all f we have $\mathrm{d}_r(v) = 0$ for all r as required.

§2. COHERENT SHEAVES

Let X be a real-analytic manifold† of dimension n, and let $\mathcal{O}$ denote the sheaf of germs of complex-valued real-analytic functions on X. Then we have the following basic facts concerning the sheaf $\mathcal{O}$.

PROPOSITION (2.1). *$\mathcal{O}$ is a coherent sheaf of rings.*

Proof. For the definition of coherence see [18, I, §2]. The problem is local, and so we may suppose X is a domain D in $\mathbf{R}^n$. If we embed $\mathbf{R}^n \subset \mathbf{C}^n$ then $\mathcal{O}$ is just the restriction to D of the sheaf of germs of holomorphic functions in $\mathbf{C}^n$. But this latter sheaf is coherent (theorem of Oka see [9, exp. XV]) and hence $\mathcal{O}$ also is coherent.

COROLLARY (2.2). *A sheaf S of $\mathcal{O}$-modules is coherent if and only if it is locally isomorphic to Coker φ, where φ is a homomorphism $\mathcal{O}^p \to \mathcal{O}^q$.*

Proof. This follows from (2.1) and [18, I, §2, Prop. 7].

PROPOSITION (2.3). *Let S be a coherent sheaf of $\mathcal{O}$-modules. Then "Theorems A and B" hold for S, i.e. for each $x \in X$ the image of $H^0(X, S)$ in S_x generates S_x as $\mathcal{O}_x$-module, and $H^q(X, S) = 0$ for $q \geqslant 1$.*

Proof. According to a result of Grauert [12], X can be real-analytically embedded in a complex manifold Y so that Y is a complexification of X and so that X has a fundamental system of neighbourhoods in Y which are Stein manifolds. The sheaf $\mathcal{O}$ is then the restriction to X of the sheaf of germs of holomorphic functions on Y. Our proposition now follows from [8, Théorème 1].

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† All manifolds are assumed to have countable topology. For connected complex manifolds this is no restriction.

COROLLARY (2.4). *Let S be a coherent sheaf of $\mathcal{O}$-modules, x a point of X. Then there exists a finite number of global sections of S whose images in S_x generate S_x as $\mathcal{O}_x$-module.*

Proof. Since S is coherent S_x is a finitely-generated $\mathcal{O}_x$-module. Let $s_1, \dots, s_n$ be a system of generators. By (2.3) each s_i can be expressed in the form

$$s_i = \sum_j a_{ij} u_{ij}$$

where $a_{ij} \in \mathcal{O}_x$ and u_{ij} is the image in S_x of a global section U_{ij}. The set of these U_{ij} is then finite and their images in S_x generate S_x.

COROLLARY (2.5). *Let S be a coherent sheaf of $\mathcal{O}$-modules, A a compact subset of X. Then there is a homomorphism $\mathcal{O}^p \to S$ which is an epimorphism at all points of A.*

Proof. For any $x \in A$ there exists, by (2.4), a finite number of global sections of S which generate S_x and hence generate S_y for all y in some neighbourhood of x [18, I, §2, Prop. 1]. The result now follows from the compactness of A.

PROPOSITION (2.6). *Let S be a coherent sheaf of $\mathcal{O}$-modules, A a compact subset of X. Then there is a sequence of coherent sheaves and homomorphisms:*

$$0 \to L_n \to L_{n-1} \to \dots \to L_0 \to S \to 0,$$

such that at any $x \in A$ this sequence is exact and each $(L_i)_x$ is free.

Proof. By (2.5) we have a homomorphism $\varphi_0 : L_0 \to S$ with $L_0 = \mathcal{O}^p$, and φ_0 an epimorphism on A. Let $S_1 = \text{Ker}\, \varphi_0$, then S_1 is also coherent [18, §2, Th. 1]. Repeating this construction with S_1 replacing S_0 we obtain in this way a sequence

$$L_{n-1} \xrightarrow{\varphi_{n-1}} L_{n-1} \to \dots \to L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} S \to 0,$$

which is exact on A and has each L_i isomorphic to $\mathcal{O}^p$ (for some p). Let $L_n = \text{Ker}\, \varphi_{n-1}$, then for any $x \in A$ we have an exact sequence of $\mathcal{O}_x$-modules:

$$0 \to (L_n)_x \to \dots \to (L_0)_x \to S_x \to 0,$$

with $(L_i)_x$ free for $0 \leqslant i \leqslant n - 1$. But S_x, as module over $\mathcal{O}_x$, has projective dimension $\leqslant n$ [11, VIII, Th. 6·5′]. Hence $(L_n)_x$ is free for $x \in A$. This completes the proof.

We now make the following observation

LEMMA (2.7). *Let L be a locally free† sheaf of $\mathcal{O}$-modules. Then L is a projective in the category of coherent sheaves of $\mathcal{O}$-modules.*

Proof. Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence of coherent sheaves. Since L is locally free we get an exact sequence of sheaves

$$0 \to Hom(L, A') \to Hom(L, A) \to Hom(L, A'') \to 0,$$

† Always assumed coherent, i.e. of finite rank.

where $Hom(L, A)$ denotes the sheaf of germs of homomorphisms $L \to A$. Using "Theorem B" (2.3) for the coherent sheaf $Hom(L, A')$ we obtain the exact sequence of groups

$$0 \to Hom(L, A') \to Hom(L, A) \to Hom(L, A'') \to 0.$$

Hence any homomorphism $L \to A''$ lifts to a homomorphism $L \to A$, proving that L is a projective.

LEMMA (2.8). *Let L_i, M_i be locally free sheaves of $\mathcal{O}$-modules and let*

$$0 \to L_n \xrightarrow{\alpha_n} L_{n-1} \to \dots \xrightarrow{\alpha_1} L_0 \xrightarrow{\alpha_0} S \to 0$$

$$0 \to M_n \xrightarrow{\beta_n} M_{n-1} \to \dots \xrightarrow{\beta_1} M_0 \xrightarrow{\beta_0} S \to 0$$

be exact sequences of sheaves. Then there is an exact sequence

$$0 \to L_n \xrightarrow{\gamma_n} \dots \xrightarrow{\gamma_2} L_1 \oplus M_2 \xrightarrow{\gamma_1} L_0 \oplus M_1 \xrightarrow{\gamma_0} M_0 \to 0$$

where $\gamma_i(x, y) = \{\alpha_i(x), \beta_{i+1}(y) + (-1)^i\theta_i(x)\}$, θ_i being a homomorphism $L_i \to M_i$ (commutating with the α_i, β_i).

Proof. This is a formal consequence of (2.7). We first extend the identity map $S \to S$ to a homomorphism $\theta_i : L_i \to M_i$ of the projective resolutions. Then we define γ_i as above and check that the resulting sequence is exact [11, IV, Ex. 3] (this construction is called the algebraic mapping cylinder).

Suppose now that X is a complex manifold, and let $\mathcal{B}$ denote the sheaf of germs of holomorphic functions on X. Then X is also a real-analytic manifold and we let $\mathcal{O}$ have the same meaning as above. If T is any sheaf of $\mathcal{B}$-modules we put $T^\circ = T \otimes_{\mathcal{B}} \mathcal{O}$. Then T° is a sheaf of $\mathcal{O}$-modules.

PROPOSITION (2.9). *If T is a coherent sheaf of $\mathcal{B}$-modules, T° is a coherent sheaf of $\mathcal{O}$-modules.*

Proof. Locally we have an exact sequence

$$\mathcal{B}^p \to \mathcal{B}^q \to T \to 0.$$

Since $\otimes$ is right-exact we get (locally) an exact sequence

$$\mathcal{O}^p \to \mathcal{O}^q \to T^\circ \to 0.$$

By (2.2) this proves that T° is coherent.

PROPOSITION (2.10). *Let Y be a closed complex analytic subspace of the complex manifold X, and let $\mathcal{B}_Y$ denote the sheaf of germs of holomorphic functions on Y (extended by zero on X–Y). Then $\mathcal{B}_Y^\circ$ is a coherent sheaf of $\mathcal{O}$-modules.*

Proof. This follows from (2.9) and the coherence of $\mathcal{B}_Y$ [9, exp. XVI].

We shall consider now a resolution which plays a fundamental role in the Riemann–Roch theorem. It is originally due to Koszul, but in the context of sheaves it was introduced by Grothendieck.

Let V be a (complex) vector space of dimension q. We denote by V^* the dual of V and by $\lambda^i(V)$ the i-th exterior power of V. Then we have the "interior product" homomorphism ($i \geqslant 1$)

$$V^* \otimes \lambda^i(V) \to \lambda^{i-1}(V)$$

given by

$$f \otimes x_1 \wedge x_2 \wedge \dots \wedge x_i \to \sum_{1 \leqslant j \leqslant i} (-1)^{j+1} f(x_j) x_1 \wedge \dots \wedge \hat{x}_j \wedge \dots \wedge x_i,$$

where $\hat{x}_j$ means that x_j is omitted.

PROPOSITION (2.11). *Let V be a complex vector space of dimension q, V^* its dual and s a non-zero element of V^*. Let $\alpha_i : \lambda^i(V) \to \lambda^{i-1}(V)$ be the interior product with s. Then we have an exact sequence*:

$$0 \to \lambda^q(V) \xrightarrow{\alpha_q} \lambda^{q-1}(V) \to \dots \xrightarrow{\alpha_1} \lambda^0(V) \to 0.$$

Proof. Choose a basis $e_1, \dots, e_q$ of V such that $s(e_1) = 1$, $s(e_i) = 0$ for $i > 1$. Then $\operatorname{Ker} \alpha_r$ and $\operatorname{Im} \alpha_{r+1}$ are both seen to be the subspaces generated by the elements

$$e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_r}$$

with $2 \leqslant i_1 < i_2 < \dots < i_r \leqslant q$. Thus the sequence is exact.

Since the interior product is functorial in V it can be defined also for vector bundles. Then (2.11) gives at once

PROPOSITION (2.12). *Let E be a continuous complex vector bundle of dimension q over a topological space X, and let s be a section of E which is nowhere zero. Then we have an exact sequence of vector bundles:*

$$0 \to E_q \xrightarrow{\alpha_q} E_{q-1} \to \dots \xrightarrow{\alpha_1} E_0 \to 0$$

where $E_i = \lambda^i(E^*)$ and α_i is given by the interior product with s.

Suppose now that E is a real-analytic complex vector bundle of dimension q over the real-analytic manifold X. Let $s : X \to E$ be a real-analytic section. Then we may introduce the *sheaf S of zeros* of s: if we identify E locally with $X \times \mathbf{C}^q$, s is given by q functions s_i: $X \to \mathbf{C}$ and we define $S = \mathcal{O}/(s_1, \dots, s_q)$. Clearly this definition is independent of the local isomorphism $E \to X \times \mathbf{C}^q$. Now put $E_i = \lambda^i(E^*)$, and let $\alpha_i : E_i \to E_{i-1}$ be the homomorphism given by the interior product with the section s of E. If we denote by L_i the locally free sheaf corresponding to E_i, and if we let $\alpha_i : L_i \to L_{i-1}$ denote also the sheaf homomorphism corresponding to $\alpha_i : E_i \to E_{i-1}$, then we have a sequence

$$0 \to L_q \xrightarrow{\alpha_q} L_{q-1} \to \dots \xrightarrow{\alpha_1} L_0 \xrightarrow{\varepsilon} S \to 0, \tag{1}$$

where $\varepsilon : L_0 \to S$ is the natural homomorphism ($L_0 = \mathcal{O}$). At points x where $s(x) \neq 0$ we have $S_x = 0$ and (1) is exact by (2.12). Suppose now that s satisfies the following property:

(P) For each x with $s(x) = 0$ there exists a local isomorphism $E \to X \times \mathbf{C}^q$, in a neighbourhood of x, so that the germ s_x is represented by germs of functions $s_i \in \mathcal{O}_x$ ($1 \leqslant i \leqslant q$) with s_i not a zero-divisor† *in $\mathcal{O}_x/(s_1, \dots, s_{i-1})$.*

† For $i = 1$ this means $s_1 \neq 0$.

Then by [11, VIII, Prop. (4.3)] (1) is exact at x. Hence we have

PROPOSITION (2.13). *Let E be a real-analytic complex vector bundle of dim q over a real-analytic manifold X. Let $s : X \to E$ be a real-analytic section satisfying property (P). Then*

$$0 \to L_q \xrightarrow{\alpha_q} L_{q-1} \to \dots \xrightarrow{\alpha_1} L_0 \xrightarrow{\varepsilon} S \to 0$$

is an exact sequence of sheaves, where S is the sheaf of zeros of s, L_i is the locally-free sheaf associated to $\lambda^i(E^*)$, α_i is given by the interior product with s and ε is the natural map $L_0 = \mathcal{O} \to S$.

§3. THE "DIFFERENCE BUNDLE"

In this section we shall give a general construction for elements of $K^*(X, Y)$ closely analogous to the "difference cochain" of classical obstruction theory.

First however we require a lemma which belongs to the general K-theory developed in [4] but is not given there explicitly.

LEMMA (3.1). *Let A, B be finite CW-complexes, A_0 a sub-complex of A. Let $a_0 \in K^*(A_0)$, $b \in K^*(B)$. Then*

$$\delta(a_0 \otimes b) = \delta(a_0) \otimes b$$

where the first δ is the coboundary homomorphism $K^*(A_0 \times B) \to K^*(A \times B, A_0 \times B)$ and the second δ is the coboundary homomorphism $K^*(A_0) \to K^*(A, A_0)$.

The proof of this lemma presents no difficulties in view of the fact that δ is induced by a map [4; §1.4] and that products are natural. The details are omitted.

COROLLARY (3.2). *Let A, A_0 be a pair of finite CW-complexes. Then*

$$\delta : K^*(A_0) \to K^*(A, A_0)$$

is a homomorphism of $K^*(A)$-modules.

Proof. This follows from (3.1) on taking $B = A$ and applying the diagonal map.

Let X be a finite CW-complex, Y a sub-complex, E, F complex vector bundles on X and let α be an isomorphism $E|Y \to F|Y$. Then to the triple (E, F, α) we shall associate an element of $K^0(X, Y)$ as follows. Let I denote the unit interval and form the sub-space

$$A = X \times 0 \cup X \times 1 \cup Y \times I$$

of $X \times I$. On A we define a complex vector bundle L by putting E on $X \times 1$, F on $X \times 0$ and using α to "join" them along $Y \times I$. More precisely let

$$\begin{aligned} I_0 &= I - \{0\}, & I_1 &= I - \{1\}, & I_{01} &= I_0 \cap I_1, \\ A_0 &= X \times 0 \cup Y \times I_1, & & & E_0 &= F, \\ A_1 &= X \times 1 \cup Y \times I_0, & & & E_1 &= E, \end{aligned}$$

and let $f_i : A_i \to X$ be induced by the projection $X \times I \to X$. Then $f_i^*(E_i)$ is a bundle on the open set A_i and α induces an isomorphism $f_0^*(E_0) \to f_1^*(E_1)$ on the open set $A_0 \cap A_1 = Y \times I_{01}$. This gives the required bundle on A. This bundle defines an element $\xi \in K^0(A)$. From the exact sequence [4, §1.4]

$$K^0(X \times I) \to K^0(A) \xrightarrow{\delta} K^1(X \times I, A)$$

we obtain $\delta\xi \in K^1(X \times I, A)$. But

$$(X \times I)/A = S(X/Y) \quad \text{and} \quad K^1(S(X/Y)) \cong K^0(X, Y) \qquad \text{(by definition)}.$$

The image of $\delta\xi$ in this isomorphism is the element we associate to (E, F, α). We shall call it the "difference bundle" of (E, F, α) and denote it by $\mathrm{d}(E, F, \alpha)$.

Remark. The universal space for K^0 is $\mathbf{Z} \times B_U$ and this is an H-space with a homotopy inverse. If we arrange for the inverse to be genuine† then $\mathrm{d}(E, F, \alpha)$ could be constructed as follows. We represent E, F by maps g, f of $X \to \mathbf{Z} \times B_U$ which agree (using α) on Y. Then gf^{-1} is a map $X/Y \to \mathbf{Z} \times B_U$ and this represents $\mathrm{d}(E, F, \alpha)$. However the formal definition we have given avoids any reference to the universal space.

We summarize the main properties of the difference bundle in the following proposition

PROPOSITION (3.3).

(i) $\mathrm{d}(E, F, \alpha)$ *is functorial, i.e. if* $f: (X', Y') \to (X, Y)$ *is a map then* $\mathrm{d}(f^*E, f^*F, f^*\alpha) = f^! \, \mathrm{d}(E, F, \alpha)$;

(ii) $\mathrm{d}(E, F, \alpha)$ *depends only on the homotopy class of* α;

(iii) *If* $Y = \emptyset$, *then* $\mathrm{d}(E, F, \alpha) = E - F$;

(iv) *If* $f^! : K^*(X, Y) \to K^*(X)$ *is the natural map, then* $f^! \, \mathrm{d}(E, F, \alpha) = E - F$;

(v) *If* α *extends to an isomorphism* $E \to F$ *on* X, *then* $\mathrm{d}(E, F, \alpha) = 0$;

(vi) $\mathrm{d}(E \oplus E', F \oplus F', \alpha \oplus \alpha') = \mathrm{d}(E, F, \alpha) + \mathrm{d}(E', F', \alpha')$;

(vii) $\mathrm{d}(F, E, \alpha^{-1}) = -\mathrm{d}(E, F, \alpha)$;

(viii) *If* D *is a vector bundle on* X, *then* $\mathrm{d}(E \otimes D, F \otimes D, \alpha \otimes 1) = \mathrm{d}(E, F, \alpha) . D$, *where on the right we use the* $K^0(X)$*-module structure of* $K^0(X, Y)$.

Proof. (i) follows at once from the construction which defines $\mathrm{d}(E, F, \alpha)$. Now let π be the projection $X \times I \to X$ and let $i_t : X \to X \times I$ be the inclusion $x \to (x, t)$. A homotopy α_t of isomorphisms $E|Y \to F|Y$ is by definition an isomorphism

$$\beta : \pi^*E|Y \times I \to \pi^*F|Y \times I.$$

Then we have

$$\begin{aligned} \mathrm{d}(E, F, \alpha_0) &= \mathrm{d}(i_0^*\pi^*E, i_0^*\pi^*F, i_0^*\beta), \\ &= i_0^! \, \mathrm{d}(\pi^*E, \pi^*F, \beta) \qquad \text{by (i).} \end{aligned}$$

Similarly $\mathrm{d}(E, F, \alpha_1) = i_1^! \, \mathrm{d}(\pi^*E, \pi^*F, \beta)$. Since $i_0 \simeq i_1$ and $K^0(X, Y)$ is an invariant of homotopy type it follows that $\mathrm{d}(E, F, \alpha_0) = \mathrm{d}(E, F, \alpha_1)$ which proves (ii). For (iii) we have to consider the homomorphism:

$$\delta : K^0(X \times S^0) \to K^1(X \times I, X \times S^0) \cong K^0(X),$$

† Since $\mathbf{Z} \times B_U$ is a loop space it can, using a construction of Milnor, be represented by a topological group.

where $S^0 = \{0\} \cup \{1\} \subset I$. Since $K^0(X \times S^0) \cong K^0(X) \otimes K^0(S^0)$ it follows, using (3.1), that we need only consider the case when X is a point, i.e.

$$\delta : K^0(S^0) \to K^1(I, S^0) \overset{\sigma}{\cong} K^0(\text{point}).$$

Since δ and the suspension isomorphism σ both commute with ch [4, §1.10] it is sufficient to consider

$$\delta : H^0(S^0) \to H^1(I, S^0) = H^0(\text{point}).$$

But $\delta(a_1) = +1$, $\delta(a_0) = -1$, where a_0, a_1 are the generators of $H^0(S^0; Z)$ corresponding to the points 0, 1. This completes the proof of (iii). From (i) and (iii) we deduce (iv). Considering the map $(X, Y) \to (X, X)$ we see that (v) follows from (i). The construction gives (vi) immediately. We turn next to the proof of (vii). Let ξ, η be the bundles on $A = X \times 0 \cup X \times 1 \cup Y \times I$ defined by (E, F, α) and (F, E, α^{-1}) respectively. Clearly $\eta \cong f^*\xi$ where $f: A \to A$ is induced by the map $x \to 1 - x$ of I. Hence $\delta\eta = g^!\delta\xi$ where $g : S(X/Y) \to S(X/Y)$ is induced by the map $x \to 1 - x$ of the suspension coordinate. But it is proved in [4, §1.5] that $g^! = -1$. Hence $\delta\eta = -\delta\xi$ and so $\mathrm{d}(F, E, \alpha^{-1}) = -\mathrm{d}(E, F, \alpha)$. Finally we observe that (viii) follows at once from (3.2) and the construction of the difference bundle.

We shall now give a generalization of the "difference bundle". Let $E_0, \dots, E_n$ be vector bundles on X, and let

(1) $$0 \to E_n \overset{\alpha_n}{\to} E_{n-1} \to \dots \to E_1 \overset{\alpha_1}{\to} E_0 \overset{\alpha_0}{\to} 0$$

be an exact sequence of vector bundles on Y (strictly speaking E_i should be replaced here by $E_i | Y$). Then we define a "generalized difference bundle",

$$\mathrm{d}(E_0, E_1, \dots, E_n; \alpha_1, \alpha_2, \dots, \alpha_n) \in K^0(X, Y)$$

as follows.

An exact sequence of the form (1) breaks up into short exact sequences

(2) $$0 \to F_r \to E_r \to F_{r-1} \to 0, \qquad 1 \leqslant r \leqslant n,$$

where $F_r = \operatorname{Ker} \alpha_r$ is a vector bundle on Y. An exact sequence of the form (2) splits (cf [2, §1]) and any two splittings are homotopic, since they differ by an element of the vector space $\operatorname{Hom}(F_{r-1}, F_r)$. Choosing one such splitting for each r we obtain isomorphisms

$$\lambda : \sum_k E_{2k+1} \to \sum_r F_r$$
$$\mu : \sum_k E_{2k} \to \sum_r F_r$$

and the homotopy classes of λ, μ are independent of the splittings. Thus $\alpha = \lambda^{-1}\mu$ is an isomorphism $\sum E_{2k} \to \sum E_{2k+1}$, whose homotopy class is independent of the splittings. Hence, by (3.3)(ii), we can define a unique element

$$\mathrm{d}(E_0, \dots, E_n; \alpha_1, \dots, \alpha_n) = \mathrm{d}(\sum E_{2k}, \sum E_{2k+1}, \alpha) \in K^0(X, Y).$$

Where no confusion can arise we write just $\mathrm{d}(E_i, \alpha_i)$.

The properties of this "generalized difference bundle" are summarized in the following proposition.

PROPOSITION (3.4).

(i) $\mathrm{d}(E_0, \dots, E_n; \alpha_1, \dots, \alpha_n)$ *is functorial* (cf. (3.3(i));

(ii) $\mathrm{d}(E_0, \dots, E_n; \alpha_1, \dots, \alpha_n)$ *depends only on the homotopy class of* $(\alpha_1, \dots, \alpha_n)$;

(iii) *If* $Y = \emptyset$, *then* $\mathrm{d}(E_0, \dots, E_n; \alpha_1, \dots, \alpha_n) = \sum_{i=0}^{n} (-1)^i E_i$;

(iv) *If* $f^! : K^0(X, Y) \to K^0(X)$ *is the natural map, then*

$$f^! \,\mathrm{d}(E_0, \dots, E_n; \alpha_1, \dots, \alpha_n) = \sum_{i=0}^{n} (-1)^i E_i;$$

(v) *If* $\alpha_1, \dots, \alpha_n$ *extend to* X *with* $0 \to E_n \xrightarrow{\alpha_n} E_{n-1} \to \dots \xrightarrow{\alpha_1} E_0 \to 0$ *exact on* X *then*

$$\mathrm{d}(E_0, \dots, E_n; \alpha_1, \dots, \alpha_n) = 0;$$

(vi) $\mathrm{d}(E_i \oplus E'_i, \alpha_i \oplus \alpha'_i) = \mathrm{d}(E_i, \alpha_i) + \mathrm{d}(E'_i, \alpha'_i)$;

(vii) $\mathrm{d}(0, E_0, \dots, E_n; 0, \alpha_1, \dots, \alpha_n) = -\mathrm{d}(E_0, \dots, E_n, \alpha_1, \dots, \alpha_n)$;

(viii) *If* D *is a vector bundle on* X, *then*

$$\mathrm{d}(E_i \otimes D, \alpha_i \otimes 1) = \mathrm{d}(E_i, \alpha_i) \cdot D.$$

Proof. (i) follows at once from the construction (cf. (3.3)(i)). In (ii) a homotopy of $(\alpha_1, \dots, \alpha_n)$ means that we have an exact sequence

$$0 \to F_n \xrightarrow{\beta_n} F_{n-1} \to \dots \xrightarrow{\beta_1} F_0 \to 0$$

on $Y \times I$, where $F_i = \pi^* E_i$ and $\pi : X \times I \to X$ is the projection. Hence a homotopy of $(\alpha_1, \dots, \alpha_n)$ induces a homotopy of the isomorphism $\alpha : \sum E_{2k} \to \sum E_{2k+1}$ of the construction above. Thus (ii) follows from (3.3)(ii). The remaining points (iii)–(viii) follow at once from the corresponding parts of (3.3).

We propose next to consider a particular difference bundle. Let X be a finite simplicial complex, E a complex vector bundle over X of dimension q. Let A, $\dot{A}$ denote the unit ball and unit sphere bundles of E. Let $\pi : A \to X$ be the projection map. Then $\pi^* E$ has a canonical section s nowhere zero on $\dot{A}$ and hence (2.12) we have an exact sequence of vector bundles on $\dot{A}$

$$0 \to F_q \xrightarrow{\alpha_q} F_{q-1} \to \dots \xrightarrow{\alpha_1} F_0 \to 0$$

where $F_i = \pi^* \lambda^i(E^*)$. Hence the element

$$\mathrm{d}(F_0, F_1, \dots, F_q; \alpha_1, \dots, \alpha_q) \in K^0(A, \dot{A})$$

is well-defined. From its definition we see that it behaves functorially.

PROPOSITION (3.5). *Let E be a complex vector bundle on the finite simplicial complex X, A, $\dot{A}$ the associated unit ball and unit sphere bundles, $\pi : A \to X$ the projection map. Let* $\xi = \mathrm{d}(\pi^* \lambda^i(E^*), \alpha_i) \in K^0(A, \dot{A})$. *Then* $\mathrm{ch}\, \xi = \varphi_* \mathfrak{T}(E)^{-1}$, where φ_* *is the Gysin homomorphism and* $\mathfrak{T}$ *is the total Todd class.*

Proof. Since $\mathrm{d}(\pi^* \lambda^i(E^*), \alpha_i)$ is functorial it is sufficient to prove the result in the universal case, i.e. we may suppose X is N-classifying for $U(q)$. Now by (3.4)(iii), if η denotes the image of ξ in $K^0(A) \cong K^0(X)$, then $\eta = \sum(-1)^i\lambda^i(E^*)$. Hence $\mathrm{ch}\,\eta = c_q(E)\cdot\mathfrak{T}(E)^{-1}$ [6, §13]. But $H^*(BU(q), BU(q-1); \mathbf{Q}) \to H^*(BU(q); \mathbf{Q})$ is an isomorphism onto the ideal generated by $c_q(E)$ and, from the definition of φ_*, $\varphi_*(x) \to c_q(E)\cdot x$ in this isomorphism. Thus (making $N \to \infty$) we have

$$\mathrm{ch}\,\xi = \varphi_*\mathfrak{T}(E)^{-1}.$$

§4. THE GROTHENDIECK ELEMENT

We are now in a position to perform our basic construction. Let X be a real-analytic manifold, $\mathcal{O}$ the sheaf of germs of complex-valued real-analytic functions on X, and let Y be any subspace of X. Then to every coherent sheaf S of $\mathcal{O}$-modules on X with support in Y we shall associate an element $\gamma_Y(S) \in \mathcal{K}^0(X, X - Y)$ in the following way.

To define $\gamma_Y(S)$ we have first (cf. §1) to define an element $f^!\gamma_Y(S) \in K^0(A, B)$ for any map $f: (A, B) \to (X, X - Y)$ with (A, B) a finite simplicial pair. Since $f(A) \subset X$ is compact we can find an open set $U \supset f(A)$ with $\bar{U}$ compact. Now apply (2.6) with $\bar{U}$ instead of A, and then restrict to U. We get an exact sequence of sheaves in U

$$0 \to L_n \xrightarrow{\alpha_n} L_{n-1} \to \dots \xrightarrow{\alpha_1} L_0 \xrightarrow{\alpha_0} S \to 0 \tag{1}$$

with all the L_i locally free.† Let E_i be the complex-vector bundle on U corresponding to L_i. E_i has a real-analytic structure but we ignore this and consider only the underlying topological structure.‡ Since the support of S is in Y we have an exact sequence of vector bundles

$$0 \to E_n \xrightarrow{\alpha_n} E_{n-1} \to \dots \xrightarrow{\alpha_1} E_0 \to 0$$

in $U - U \cap Y$. Since $f(B) \subset U - U \cap Y$ it follows that we have an induced exact sequence on B. We then define

$$f^!\gamma_Y(S) = \mathrm{d}(f^*E_0, \dots, f^*E_n; f^*\alpha_1, \dots, f^*\alpha_n),$$

where the right hand side in the "generalized difference bundle" defined in §3. We will first show that $f^!\gamma_Y(S)$ is independent of the choice of resolution (1). Suppose therefore

$$0 \to M_n \xrightarrow{\beta_n} M_{n-1} \to \dots \xrightarrow{\beta_1} M_0 \xrightarrow{\beta_0} S \to 0$$

is another resolution in U. Then applying (2.8), with U instead of X, we get an exact sequence of locally free sheaves on U

$$0 \to L_n \xrightarrow{\gamma_n} \dots \to L_1 \oplus M_2 \xrightarrow{\gamma_1} L_0 \oplus M_1 \xrightarrow{\gamma_0} M_0 \to 0$$

where
$$\gamma_i(x, y) = \{\alpha_i(x), \beta_{i+1}(y) + (-1)^i\theta_i(x)\}.$$

Passing to the corresponding vector bundles we get an exact sequence

$$0 \to E_n \xrightarrow{\gamma_n} \dots \to E_1 \oplus F_2 \xrightarrow{\gamma_1} E_0 \oplus F_1 \xrightarrow{\gamma_0} E_0 \to 0,$$

† As remarked in [18, II, §4] a coherent sheaf L is locally free if and only if L_x is free for all x.

‡ In fact according to deep results of Grauert the topological structure determines the analytic structure up to isomorphism.

where γ_i is given by the same formula. Since the γ_i are defined in all of U it follows from (3.4)(v) that, applying f^*,

(2) $$d(f^*E_i \oplus f^*F_{i+1}, f^*\gamma_i) = 0.$$

On the other hand, in $U - Y \cap U$, we have a homotopy of γ_i with $\alpha_i \oplus \beta_{i+1}$ obtained by defining

$$\gamma_i(x, y; t) = \{\alpha_i(x), \beta_{i+1}(y) + (-1)^i t \cdot \theta_i(x)\};$$

the fact that, for all t, this preserves exactness is easily checked; for example it follows from the same formal result as (2.8), taking $S = 0$ and replacing θ_i by $t\theta_i$. Hence from (3.4)(ii) we have

(3) $$d(f^*E_i \oplus f^*F_{i+1}, f^*\gamma_i) = d(f^*E_i \oplus f^*F_{i+1}, f^*\alpha_i \oplus f^*\beta_{i+1}).$$

Then from (2), (3) and (3.4)(vi), (vii) it follows that

$$d(f^*E_i, f^*\alpha_i) = d(f^*F_i, f^*\beta_i),$$

showing that our definition of $f^!\gamma_Y(S)$ is certainly independent of the resolution (1).

Next we observe that $f^!\gamma_Y(S)$ does not depend on the choice of U. In fact if $V \supset U$ is an open set with $\bar{V}$ compact, then a resolution of S in V restricts to give a resolution in U. Then using this resolution to define $f^!\gamma_Y(S)$ we see at once that it is the same whether we use U or V.

The naturality of $f^!\gamma_Y(S)$ follows from (3.4)(i). It remains therefore to prove that it depends only on the homotopy class of f. Suppose therefore $F: (A \times I, B \times I) \to (X, X - Y)$ is a homotopy. Choose U open containing $F(A \times I)$ and with $\bar{U}$ compact. The result now follows from (3.4)(ii).

The element $\gamma_Y(S)$ will be called the *Grothendieck element* of S. It plays an important role in the Riemann–Roch theorem. We shall now compute $\gamma_Y(S)$ in the simplest, but most important, case.

PROPOSITION (4.1). *Let X denote the domain $\sum_1^n |z_i|^2 < 1$ in $\mathbf{C}^n$, Y the sub-space of X given by $z_i = 0$ $(1 \leqslant i \leqslant q)$. Let $\mathscr{B}_Y$ denote the sheaf of holomorphic functions on Y (zero on $X - Y$), $\mathscr{B}_Y^\circ = \mathscr{B}_Y \otimes_{\mathscr{B}_X} \mathscr{O}_X$ ($\mathscr{O}_X$ = sheaf of real-analytic germs $X \to \mathbf{C}$). Then*

$$\operatorname{ch} \gamma_Y(\mathscr{B}_Y^\circ) = u,$$

where u is the generator of $\mathscr{H}^{2q}(X, X - Y; \mathbf{Z})$ corresponding to Y.

Proof. Since $\mathscr{B}_Y^\circ = \mathscr{O}_X/(z_1, \dots, z_q)$ we can apply (2.11) with E the trivial bundle $X \times \mathbf{C}^q$. This gives us a resolution of $\mathscr{B}_Y^\circ$. Restricting this resolution to the ball A given by

$$\sum_{i=1}^q |z_i|^2 \leqslant 1/2, \qquad z_i = 0 (i > q),$$

and its boundary $\dot{A}$, and applying (3.5) we get

$$\operatorname{ch} \gamma_Y(\mathscr{B}_Y^\circ) = u,$$

where $u \in H^{2q}(A, \dot{A}; \mathbf{Z})$ is the generator corresponding (or dual) to Y. Since

$$H^{2q}(A, \dot{A}; \mathbf{Z}) \cong \mathscr{H}^{2q}(X, X - Y; \mathbf{Z})$$

this concludes the proof.

Remark. Applying (4.1) with $n = q$ we get a sheaf-theoretical construction for the Bott generator of $K^0(S^{2q}, \text{point})$.

§5. ANALYTIC CYCLES

The purpose of this section is to overcome the technical difficulties, referred to in the Introduction, concerning the definition of complex analytic homology and cohomology classes. We should perhaps point out that, for our purposes, it is sufficient to consider complex analytic *subspaces* of a complex manifold: abstract complex spaces are not needed. This is a simplification which enables us to get by with an elementary treatment.

We start by giving a proof of the following "well-known result".

LEMMA (5.1). *Let X be a differentiable manifold, Y a closed submanifold of codimension k. Let $f : A \to X$ be a map of a finite simplicial complex A of dimension $<k$ into X. Then f is homotopic, by an arbitrarily small homotopy, to a map g with $g(A) \subset X - Y$.*

Proof. First we prove the lemma in the case when X is an open cube in $\mathbf{R}^n$ ($|x_i| < 1$, $1 \leqslant i \leqslant n$) and Y is given by $x_i = 0$, $1 \leqslant i \leqslant k$. Then it is trivial to construct a triangulation of X, of arbitrarily fine mesh, and such that $Y \cap X^{k-1} = \varnothing$, where X^{k-1} is the $(k-1)$-skeleton of the triangulation. Now apply the simplicial approximation theorem and the result follows. In the general case we can take a sufficiently fine subdivision of A to ensure that, for each simplex σ of A, we have $f(\sigma) \subset U_\sigma$, where U_σ is an open co-ordinate cube ($|x_i| < 1$) and $Y \cap U_\sigma = \varnothing$ or $Y \cap U_\sigma$ is given by $x_i = 0$, $1 \leqslant i \leqslant k$. We now use the lemma for U_σ (as just proved) and push the simplices off Y one at a time. At each stage we choose all the homotopies f_t so small that $f_t(\sigma) \subset U_\sigma$ (for all σ) so that no change of co-ordinate cubes is necessary. The homotopy extension property of subcomplexes has to be invoked in the form: a "small" homotopy of f on a subcomplex extends to a "small" homotopy on the whole complex.

LEMMA (5.2). *Let X be a complex manifold, Y a closed complex analytic subspace of complex codimension q. Let (A, B) be a finite simplicial pair of dimension $< 2q$. Then any map $f : (A, B) \to (X, X - Y)$ is homotopic to a map g with $g(A) \subset X - Y$.*

Proof. There is a sequence of closed subspaces of Y:

$$\varnothing = Y_0 \subset Y_1 \subset \ldots \subset Y_n = Y$$

such that $Y_i - Y_{i-1}$ is a closed submanifold of $X - Y_{i-1}$. (5.2) now follows by a repeated application of (5.1): since all homotopies are small we can ensure $f_t(B) \subset X - Y$ throughout the homotopy.

LEMMA (5.3). *Let X, Y be as in* (5.2). *Then $i : X - Y \to X$ induces isomorphisms*

$$i^* : \mathscr{H}^r(X; \mathbf{Z}) \to \mathscr{H}^r(X - Y; \mathbf{Z}) \qquad 0 \leqslant r \leqslant 2q - 2.$$

Proof. Let A be any finite simplicial complex, $f : A \to X$ a map. Applying (5.2) to the pair $(A^{2q-1}, \varnothing)$, where A^{2q-1} is the $(2q-1)$-skeleton of A, we see that $f \simeq g$ with $g(A^{2q-1}) \subset X - Y$. This shows that i^* is a monomorphism. Now apply (5.2) to the pair $(A^{2q-2} \times I, \; A^{2q-2} \times 0 \cup A^{2q-2} \times 1)$ and we see that the homotopy class of $g : A^{2q-2} \to X - Y$ is uniquely determined by f. This shows that i^* is an epimorphism.

Using (5.3) we can now define the class dual to a complex subspace. Let X be a connected complex manifold, Y a closed complex analytic subspace of complex codimension q. Let W be the singular subspace of Y, so that the complex codimension of W in X is $\geqslant q+1$. Then by (5.3) applied to X, W we get an isomorphism

$$i^* : \mathscr{H}^{2q}(X; \mathbf{Z}) \to \mathscr{H}^{2q}(X - W; \mathbf{Z}).$$

But $Y - W$ is a closed submanifold of $X - W$, and both are naturally oriented by the complex structure. Thus $Y - W$ defines a class $y'' \in H^{2q}(X - W; \mathbf{Z})$ and hence a class

$$y' \in \mathscr{H}^{2q}(X - W; \mathbf{Z}).$$

Then, in view of the fact that i^* is an isomorphism, we obtain

Proposition (5.4). *Let X be a complex manifold, Y a closed complex analytic subspace of complex codimension q, and let W be the singular subspace of Y. Then there is one and only one class $y \in \mathscr{H}^{2q}(X; \mathbf{Z})$ whose image in $\mathscr{H}^{2q}(X - W; \mathbf{Z})$ is the class y' defined by the closed submanifold $Y - W$ of $X - W$.*

The class y of (5.4) will be referred to as the class defined by Y.

Lemma (5.5). *Let X, Y, W be as in (5.4). Then $j : (X - W, X - Y) \to (X, X - Y)$ induces isomorphisms*

$$j^* : \mathscr{H}^r(X, X - Y; \mathbf{Z}) \to \mathscr{H}^r(X - W, X - Y; \mathbf{Z}) \qquad (0 \leqslant r \leqslant 2q).$$

Proof. This is quite similar to the proof of (5.3). Let $f : (A, B) \to (X, X - Y)$ be a map with A, B a simplicial pair. Applying (5.2) to the pair (A^{2q+1}, B^{2q+1}) and the spaces X, $X - W$ we see that $f \cong g : (A, B) \to (X, X - Y)$ with $g(A^{2q+1}) \subset X - W$. This shows that j^* is a monomorphism. Applying (5.2) to the pair $(A^{2q} \times I, B^{2q} \times I \cup A^{2q} \times 0 \cup A^{2q} \times 1)$ and the spaces X, $X - W$ we deduce as in (5.3) that j^* is an epimorphism.

Lemma (5.6). *Let X, Y be as in (5.2) with Y irreducible. Then $\mathscr{H}^{2q}(X, X - Y; \mathbf{Z})$ is an infinite cyclic group, generated by an element u whose image in $\mathscr{H}^{2q}(X; \mathbf{Z})$ is the element y defined by Y.*

Proof. By (5.3) and (5.5) we have a commutative diagram

$$\begin{array}{ccc} \mathscr{H}^{2q}(X, X - Y; \mathbf{Z}) & \xrightarrow{j^*} & \mathscr{H}^{2q}(X - W, X - Y; \mathbf{Z}) \\ \downarrow & & \downarrow \\ \mathscr{H}^{2q}(X; \mathbf{Z}) & \xrightarrow{i^*} & \mathscr{H}^{2q}(X - W; \mathbf{Z}) \end{array}$$

with i^* and j^* both isomorphisms. In view of the definition of y this shows that for the proof of the lemma we can replace X, Y by $X - W$, $Y - W$ i.e. we can suppose Y non-singular and connected (since it is irreducible).

With this assumption consider the commutative diagram

$$\begin{array}{ccc} H^{2q}(X, X - Y; \mathbf{Z}) & \xrightarrow{\alpha} & \mathscr{H}^{2q}(X, X - Y; \mathbf{Z}) \\ \downarrow \beta & & \downarrow \gamma \\ H^{2q}(X; \mathbf{Z}) & \xrightarrow{\delta} & \mathscr{H}^{2q}(X; \mathbf{Z}) \end{array}$$

As pointed out in §1, α is an epimorphism. Now $H^{2q}(X, X - Y; \mathbf{Z})$ is infinite cyclic, generated by an element u' with $\beta(u')$ the class defined by Y. Hence $\mathscr{H}^{2q}(X, X-Y; \mathbf{Z})$ is generated by $u = \alpha(u')$ where $\gamma(u) = \delta\beta(u') = y$. It only remains to check that u has infinite order, and this follows at once on considering the restriction of u to a small neighbourhood of a point on Y.

The generator u of $\mathscr{H}^{2q}(X, X - Y; \mathbf{Z})$ will be called the canonical generator.

§6. THE MAIN THEOREMS

Let X be a complex manifold and let $y \in \mathscr{H}^{2q}(X; \mathbf{Z})$. We shall say that y is *complex analytic* if there exist closed irreducible complex analytic subspaces Y_i of X $(1 \leqslant i \leqslant k)$ and integers n_i so that $y = \sum_1^k n_i y_i$, where y_i is the element of $\mathscr{H}^{2q}(X; \mathbf{Z})$ defined by Y_i (cf. §5). Then we may state our main theorem.

THEOREM (6.1). *Let X be a complex manifold, and let $y \in \mathscr{H}^{2q}(X; \mathbf{Z})$ be complex analytic. Then* $\mathrm{d}_r y = 0$ *for all r, where the* d_r *are the differentials of the spectral sequence* $H^* \Rightarrow K^*$ (cf. *Introduction*). *In particular for each prime p,* $\delta_p \mathscr{P}_p^1(y) = 0$.

Remark. $\mathscr{P}_p^1$ is considered here as a homomorphism $\mathscr{H}^{2q}(X, \mathbf{Z}) \to \mathscr{H}^{2q+2p-2}(X; \mathbf{Z}_p)$ and, for $p = 2$, $\delta_p \mathscr{P}_p^1$ is to be interpreted as Sq^3.

In view of (1.3), (5.2), (5.6) and (7.1) this theorem will follow from:

PROPOSITION (6.2). *Let X be a complex manifold, Y a closed irreducible complex analytic subspace of complex codimension q, u the canonical generator of $\mathscr{H}^{2q}(X, X-Y; \mathbf{Z})$. Then*

$$\mathrm{ch}\, \gamma_Y(\mathscr{B}_Y^\circ) = \rho_*(u) + \textit{higher terms},$$

where $\gamma_Y(\mathscr{B}_Y^\circ)$ is the Grothendieck element of $\mathscr{B}_Y^\circ$ and ρ_* is induced by the coefficient homomorphism $\rho : \mathbf{Z} \to Q$.

$\mathscr{B}_Y^\circ$ is coherent by (2.10) so that $\gamma_Y(\mathscr{B}_Y^\circ)$ is defined (§4). Since $\gamma_Y(\mathscr{B}_Y^\circ)$ and u are both functorial, and since u generates $\mathscr{H}^{2q}(X, X-Y; \mathbf{Z})$ (5.6) it is sufficient to prove (6.2) for the special case when X is the domain $\sum_1^n |z_i|^2 < 1$ and Y is the subspace $z_i = 0$ $(1 \leqslant i \leqslant q)$ But this case has been proved in (4.1). Thus (6.2) and hence (6.1) are proved.

Remark.† If X has the homotopy type of a finite CW-complex, then $\mathscr{H}$ in (6.1) can be replaced by the singular cohomology H. For example if X is *compact* then *either* from the triangulability of X *or* from Morse Theory it follows that X has the homotopy type of a finite CW-complex, so that $H^*(X) \cong \mathscr{H}^*(X)$.

Theorem (6.1) gives necessary conditions on a class $y \in \mathscr{H}^{2q}(X; \mathbf{Z})$ in order that it should be complex analytic. In order to show that these conditions are not vacuous we have still to exhibit examples of complex manifolds X with classes y such that $\mathrm{d}_r(y) \neq 0$. This is our next task, and we shall actually construct examples in two restricted classes of complex manifolds, namely projective algebraic manifolds and Stein manifolds. This will show that (6.1) gives non-trivial conditions even in these more restrictive classes.

First we deal with Stein manifolds.

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† See note added in proof on p. 45.

THEOREM (6.3). *For any prime p there exists a Stein manifold X, and a cohomology class $y \in H^{2q}(X; \mathbf{Z})$ with $\delta_p \mathscr{P}_p^1 y \neq 0$. This class is not complex analytic.*

Since $\delta_p \mathscr{P}_p^1 \neq 0$ in the Steenrod algebra this will follow from (6.1) and the following general result.

PROPOSITION (6.4). *Let A be a finite polyhedron. Then there exists a Stein manifold X, having the homotopy type of a finite polyhedron, and having A as a retract. In particular $H^*(A; \mathbf{Z})$ is a direct factor (as module over the Steenrod algebra) of $H^*(X; \mathbf{Z})$.*

Proof. By a construction of Thom [21, III, §2] we can embed A in a compact C^∞-differentiable manifold B so that A is a retract of B. By a theorem of Whitney [22] B can be given a real-analytic structure. By results of Grauert [12] we can then embed B in a complexification X which is a Stein manifold, has B as a retract and is the interior of a compact manifold $\bar{X}$ with regular boundary. Thus $\bar{X}$ and X have the homotopy type of a finite polyhedron (cf. the *Remark* after (6.2)). This completes the proof.

Next, we consider projective algebraic manifolds.

THEOREM (6.5). *For any prime p there exists a projective algebraic manifold X and a cohomology class $y \in H^{2q}(X; \mathbf{Z})$ such that*

(i) $\delta_p \mathscr{P}_p^1(y) \neq 0$,

(ii) *y is of order p.*

This class is not complex analytic.

This will follow from (6.1) and the following general results.

PROPOSITION (6.6). *Let G be any finite group and n any positive integer ($n > 2$). Then there exists a projective algebraic manifold X having the same n-type as the product of Eilenberg–Maclane spaces $K(\mathbf{Z}, 2) \times K(G, 1)$. In particular $H^*(G; \mathbf{Z})$ (up to dimension n) is a direct factor of $H^*(X; \mathbf{Z})$.*

PROPOSITION (6.7). *Let p be any prime, then there exists a finite group G and a cohomology class $y \in H^{2q}(G; \mathbf{Z})$ of order p with $\delta_p \mathscr{P}_p^1(y) \neq 0$.*

Remark. Proposition (6.6) and its proof which we give below, are due to J-P. Serre. More generally Serre has remarked that the construction of [19, §20], but with a projective representation instead of a linear one, gives a projective algebraic manifold X having $\pi_i = 0$, $(3 \leq i \leq n)$, $\pi_2 = \mathbf{Z}$, $\pi_1 = G$ (operating trivially on π_2) and with any given k-invariant $k \in H^3(G; \mathbf{Z})$. The case (6.6) corresponds to the case $k = 0$.

Proof of (6.6). In [19, §20] Serre showed that given $r \geqslant 1$ one can find a representation of G in $\mathbf{C}^{N+1}$ and an algebraic manifold Y in P^N invariant under the operation of G (induced from the operation on $\mathbf{C}^{N+1}$) such that:

(i) G operates without fixed points on Y;

(ii) Y is the complete intersection of a number of hypersurfaces of P_N of degree d which are non-singular on Y and intersect transversally;

(iii) $\dim_{\mathbf{C}} Y = r$,

and that then Y is necessarily connected and $X = Y/G$ is a projective algebraic manifold. We observe first that $Y \to P_N$ is a $(r-1)$-homotopy equivalence. To see this consider the embedding $P_N \to P_M$ given by all polynomials of degree d. Then by (ii) Y is the intersection of P_N with a linear subspace L of P_M, the intersection being transversal. Now it is easy to show by a differential-geometric argument that the map $L \cap P_N \to P_N$ is homotopically equivalent to $L' \cap P_N \to P_N$, where L' is any other linear subspace of the same dimension which intersects P_N transversally. Thus $Y \to P_N$ may be replaced by $Y' \to P_N$, where Y' is a complete intersection of non-singular hypersurfaces of P_N which meet transversally. But now we can apply the Lefschetz theorem, in the form given by Bott [7], using induction on the number of hypersurfaces. Now let $v \in H^2(Y; \mathbf{Z})$ be the class induced by the canonical generator of $P_N: -v$ may be considered as the Chern class of the bundle on Y induced from the line bundle $\mathbf{C}^{N+1} - \{0\} \to P_N$. Since G operates on this whole bundle it follows that there is a line bundle ξ on X such that $\eta = \pi^*\xi$, where $\pi : Y \to X$ is the covering map. Then $v = \pi^*u$, where u is the Chern class of ξ. *Let* $f: X \to K(\mathbf{Z}, 2)$ be a map representing u; $g: X \to B_G$ a map inducing the covering $Y \to X$, $\bar{g}: Y \to E_G$ the covering map of g. Then $(f \circ \pi, \bar{g}): Y \to K(\mathbf{Z}, 2) \times E_G$ is the covering map of $(f, g): X \to K(\mathbf{Z}, 2) \times B_G$. By what we have shown above $(f\circ\pi, \bar{g})$ is a $(r-1)$-homotopy equivalence. Hence (f, g) is a $(r-1)$-homotopy equivalence. Taking $r - 1 \geqslant n$, and recalling that B_G is a $K(G, 1)$ this completes the proof of (6.6).

Proof of (6.7). Consider first the case when p is odd. We take $G = \mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_p$. Since this group has exponent p it follows that every non-zero element of $H^q(G; \mathbf{Z})$ $(q \geqslant 1)$ has order p. Hence from the exact cohomology sequence of the coefficient sequence

$$0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_p \to 0,$$

we see that we may identify $H^*(G; \mathbf{Z})$ with the kernel of the Bockstein homomorphism

$$\beta : H^*(G; \mathbf{Z}_p) \to H^*(G; \mathbf{Z}_p).$$

Now, by the Künneth formula over a field, we have

$$H^*(G; \mathbf{Z}_p) = \mathbf{Z}_p[u_1, u_2, u_3, v_1, v_2, v_3]$$

where $u_i^2 = 0$, $i = 1, 2, 3$. Moreover we have $\beta(u_i) = v_i$, $\mathscr{P}^1(u_i) = 0$, $\mathscr{P}^1(v_i) = v_i^p$. If we put $\mathscr{Q}^1 = \beta\mathscr{P}^1 - \mathscr{P}^1\beta$ then $\mathscr{Q}^1$ is an anti-derivation and for $x \in \operatorname{Ker}\beta$ we clearly have $\mathscr{Q}^1(x) = \beta\mathscr{P}^1(x)$. Hence to prove (6.7) we have only to find $y \in H^{2q}(G; \mathbf{Z}_p)$ such that $\beta(y) = 0$, $\mathscr{Q}^1(y) \neq 0$. An element y with this property is $y = \beta(u_1u_2u_3)$. In fact $\beta y = 0$ trivially and

$$\begin{aligned}\mathscr{Q}^1(y) &= \mathscr{Q}^1 \textstyle\sum v_1u_2u_3,\\ &= -\textstyle\sum(v_1v_2{}^pu_3 - v_1u_2v_3{}^p),\\ &\neq 0,\end{aligned}$$

where each $\sum$ is summed over the three cyclic permutations of the suffixes 1, 2, 3.

For the case $p = 2$ the same element works, with the understanding now that $v_i = u_i^2$.

Remarks. (1) The element y constructed in the proof of (6.5) was of dimension 4. A closer examination of our construction shows that, for $p = 2$, we can take X in (6.5) to have (complex) dimension 7. In (6.3) we can also take y to have dimension 4 and, for $p = 2$, $\dim_{\mathbf{C}} X = 8$.

(2) It is known [9] that every 2-dimensional integral cohomology class on a Stein manifold is complex analytic. (6.3) shows this does not extend to cohomology classes of higher dimension.

(3) The classical theorem of Lefschetz–Hodge [14] implies that every element of finite order in $H^2(X;\ \mathbf{Z})$ (X a projective algebraic manifold) is complex analytic (i.e. algebraic). (6.5) shows therefore that Hodge's conjecture [13] is not true for cohomology classes of higher dimension.

(4) The preceding remarks involve torsion in an essential manner. It seems likely that the answers would be quite different for $H^*(X;\ \mathbf{Q})$.

§7. THE OPERATORS d_{2p-1}

In this section we shall give a proof of the statement concerning the operators d_{2p-1} of the spectral sequence $H^* \Rightarrow K^*$, which was used without proof in §6. More comprehensive results in this direction will be found in [1], although the formulation there is in terms of the Postnikov system of B_U and not in terms of the spectral sequence.

The result used in §6 is:

PROPOSITION (7.1). *If* $\mathrm{d}_r u = 0$ *for all* r, *where the* d_r *are the operators of the spectral sequence* $H^*(X;\ \mathbf{Z}) \Rightarrow K^*(X)$, *then* $\delta_p \mathscr{P}_p^1(u) = 0$.

This will follow from the following more precise result:

PROPOSITION (7.2). *Let* $u \in H^k(X;\ \mathbf{Z})$, *p a prime, then there exists an integer N prime to p such that* $\mathrm{d}_s(Nu) = 0$, $s < 2p - 1$, *and* $\mathrm{d}_{2p-1}(Nu) = -N\delta_p\mathscr{P}_p^1(u)$.

Remark. Taking $p = 2$ it follows that d^3 coincides with the primary operation Sq^3. In general, let C be the class of abelian groups in which every element has finite order prime to p. Then (7.2) asserts that the term E_r in the spectral sequence is isomorphic (mod C) to $H^*(X;\ \mathbf{Z})$ for $r \leqslant 2p - 1$, and that on E_{2p-1} the boundary d_{2p-1} coincides (mod C) with $-\delta_p\mathscr{P}_p^1$.

In (1.2) we gave conditions for the vanishing of the operators d_r in terms of the Chern character. In the same direction we can give the following description of the first non-vanishing d_r (cf. [4; §2]).

LEMMA (7.3). *In the notation of* (1.2) *suppose that* $\mathrm{d}_s v = 0$ *for all* $s < r$ *and that* α *is a cochain representative for* $(\mathrm{ch}\ \xi)_{k+r-1}$. *Then* $\delta\alpha$ *is an integral cochain and is a representative for* $\mathrm{d}_r v$.

For the proof of (7.2) we shall require the following properties of the Eilenberg–MacLane spaces $K(\mathbf{Z}, n)$ (cf. [10]):

(1) $H^{n+q}(K(\mathbf{Z}, n);\ \mathbf{Z})$ is finite and independent of n for $0 < q < n$;

(2) The p-primary part of $H^{n+q}(K(\mathbf{Z}, n);\ \mathbf{Z})$ is zero for $0 < q < 2p - 1 \leq n$;

(3) If $n > 2p - 1$, the p-primary part of $H^{n+2p-1}(K(\mathbf{Z}, n);\ \mathbf{Z})$ is cyclic of order p with generator $\delta_p\mathscr{P}_p^1(v)$ where v is the fundamental class.

We shall also need the corresponding results on the stable homotopy of spheres (c.f. [17, §6, Prop. 11]);

(4) $\pi_q^S = \pi_{n+q}(S^n)$ is finite and independent of n for $0 < q < n - 1$;

(5) The p-primary part of π_q^S is zero for $0 < q < 2p - 3$.

(6) The p-primary part of π_{2p-3}^S is cyclic of order p.

Using these we shall first prove an auxiliary lemma.

LEMMA (7.4). *Let p be a prime and let $P_n(\mathbf{C})$ denote complex projective space of dimension n. Then there exists an S-map $g : S^{2p} \to P_p(\mathbf{C})$ of degree Mp where M is prime to p.*

Proof. We consider the exact stable homotopy sequence of the pair $P_n(\mathbf{C})$, $P_{n-1}(\mathbf{C})$:

$$\to \pi_q^S(P_{n-1}(\mathbf{C})) \xrightarrow{i_*} \pi_q^S(P_n(\mathbf{C})) \xrightarrow{j_*} \pi_{q-2n}^S \xrightarrow{\delta}$$

Here we have written $\pi_q^S(X)$ for the stable group $\pi_{n+q}(S^n(X))$, and we have used the isomorphism [5; §7, Th. II]

$$\pi_q^S(X, Y) \cong \pi_q^S(X/Y).$$

To prove the lemma we have to show the existence of an element $\alpha \in \pi_{2p}^S(P_p(\mathbf{C}))$ with $j_*(\alpha) = Mp\eta$, where η is the generator of $\pi_0^S \cong \mathbf{Z}$. This is equivalent to showing that $\delta(\eta) \in \pi_{2p-1}^S(P_{p-1}(\mathbf{C}))$ has order dividing Mp. But putting $q = 2p - 1$, $n = 2, \dots, p - 1$ in the exact sequence and using (5) and (6) we deduce that the p-primary part of $\pi_{2p-1}^S(P_{p-1}(\mathbf{C}))$ is cyclic of order p. This completes the proof.

We turn now to the proof of (7.2). Since the spectral sequence is natural it is sufficient to take X as a large finite skeleton of $K(\mathbf{Z}, k)$ and u as the fundamental class ι. Then from (1)–(3) we deduce the existence of an integer N prime to p such that $d_s(N\iota) = 0$ for $s < 2p - 1$ and

$$d_{2p-1}(N\upsilon) = \lambda N \delta_p \mathscr{P}_p^1(\upsilon),$$

for some $\lambda \in \mathbf{Z}_p$. Since moreover the spectral sequence is stable under suspension it follows that λ is independent of k. To determine the value of λ we take an example space

$$X = S^{2n}(P_p(\mathbf{C})) \cup E^{2n+2p+1},$$

where the cell $E^{2n+2p+1}$ is attached by a map g of degree Mp as given by (7.4). We consider $u = \sigma^{2n}(x) \in H^{2n+2}(X; \mathbf{Z})$ where x is the generator of $H^2(P_p(\mathbf{C}); \mathbf{Z})$ and σ^{2n} is the suspension isomorphism. Now we have an element $\xi \in K^*(P_p(\mathbf{C}))$ with

$$\text{ch}\xi = e^x - 1 = x + \frac{x^2}{2!} + \dots + \frac{x^p}{p!}$$

and hence an element $\eta = \sigma^{2n}(\xi) \in K^*(S^{2n}(P_p(\mathbf{C})))$ with

$$\text{ch}\eta = \sigma^{2n}(e^x - 1) = u + \dots + \frac{\sigma^{2n}(x^p)}{p!}.$$

Then by (7.3) we see that $d_r u$ is represented by the cochain $My/(p - 1)!$ where y if the generator of the cochain group $C^{2n+2p+1}(X; \mathbf{Z})$. On the other hand in $P_p(\mathbf{C})$ we have

$$\mathscr{P}_p^1(x) = \bar{x}^p,$$

where $\bar{x}$ is the mod p reduction of x, and hence $\delta_p \mathscr{P}_p^1(u)$ is represented by the cochain My. Since $(p - 1)! \equiv -1 \bmod p$ this shows that $\lambda = -1$, and completes the proof of (7.2).

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Note added in proof. Douady has shown (*Séminaire Bourbaki*, December 1961, No. 223) that one can avoid $\mathscr{H}$ and that our main result (6.1) holds also for ordinary singular cohomology.