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ELLIPTIC GENERA, INVOLUTIONS, AND HOMOGENEOUS SPIN MANIFOLDS

Dedicated to Jacques Tits on the occasion of his sixtieth birthday

ABSTRACT. We study the normalized elliptic genera $\Phi(X) = \varphi(X)/\varepsilon^{k/2}$ for $4k$-dimensional homogeneous spin manifolds X and show that they are constant as modular functions. The basic tool is a reduction formula relating $\Phi(X)$ to that of the self-intersection of the fixed point set of an involution γ on X. When $\Phi(X)$ is a constant it equals the signature of X. We derive a general formula for sign(G/H), $G \supset H$ compact Lie groups, and determine its value in some cases by making use of the theory of involutions in compact Lie groups.

0. INTRODUCTION

A genus in the sense of [14] is a homomorphism of the oriented bordism ring Ω (of oriented, compact, differentiable manifolds) into a ring with unit. Well-known examples are the signature and the $\hat{A}$-genus. The elliptic genus φ, recently introduced by Ochanine (see [22], [20]), is a homomorphism of the graded ring Ω to the graded ring of modular forms for the subgroup $\Gamma_0(2)$ of $SL_2(\mathbb{Z})$. As a general reference for the theory of elliptic genera we recommend the proceedings of the Princeton conference of 1986 [1] edited by Landweber. There one also finds the survey article [20] which gives a historical introduction to the subject.

On the complex projective spaces $P_{2k}(\mathbb{C})$, the elliptic genus takes values given by the formula

$$\sum_{k=0}^{\infty} \varphi(P_{2k}(\mathbb{C}))t^{2k} = (1 - 2\delta t^2 + \varepsilon t^4)^{-1/2}$$

where δ and ε are modular forms of weights 2 and 4 respectively. The elliptic genus of a $4k$-dimensional manifold X is a modular form of weight $2k$. We shall study the genus $\Phi = \varphi/\varepsilon^{k/2}$ which (for k even) is a modular function for $\Gamma_0(2)$. It will also be called elliptic genus. According to Witten [28, p. 173], the modular function $\Phi(X)$ has (in one of the two cusps of $\Gamma_0(2)$) a q-development which can be regarded as the equivariant signature of the free loop space of X with respect to the natural action of S^1 on the loop space. We take this q-development (see 1.2(5)) as definition of $\Phi(X)$. The coefficients of this q-

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development are twisted signatures (cf. [4, §6]), namely indices of the signature operator twisted by complex vector bundles associated to the tangent bundle of X. The constant term of the q-series is the usual signature $\operatorname{sign}(X)$ of X.

If a compact Lie group acts on X, then the equivariant genus $\Phi(X)_g$ is defined for any $g \in G$. The coefficients of its q-development are now equivariant twisted signatures (see 1.4). According to the rigidity theorem of Witten–Taubes–Bott [10] the value $\Phi(X)_g$ does not depend on g provided X is a spin manifold and G is connected.

Let now X be a homogeneous space G/H where G and H are compact Lie groups (G connected). Let $\mathbb{H}$ denote the upper half plane and put $q = e^{2\pi i\tau}$ for $\tau \in \mathbb{H}$. We may then view the equivariant elliptic genus as a function on $G \times \mathbb{H}$. If $X = G/H$ is a spin manifold the rigidity theorem tells us that this function does not depend on $g \in G$. Our main result (Theorem 2.3) will show that it also does not depend on $\tau \in \mathbb{H}$ in this case, i.e. all coefficients of q^n ($n > 0$) vanish. Since the constant term of $\Phi(X)$ equals the signature, we thus get

$$\Phi(X) = \operatorname{sign}(X)$$

(for all homogeneous spin manifolds X). If $X = G/H$ is not a spin manifold, then in general both independence statements are wrong. If, for example, $X = P_{2k}(\mathbb{C})$, then $\Phi(X)_g$ depends non-trivially on g and τ.

The basic tool for our study is the formula

$$\Phi(X)_g = \Phi(X^g \circ X^g)$$

for an orientation-preserving involution g on an oriented differentiable manifold X. Here $X^g \circ X^g$ denotes a transversal self-intersection of the fixed point set X^g. By taking constant terms in the q-developments this gives back an old formula for the signature (see 1.4 (10) and [15], [4, Prop. 6.15]). A consequence of our formula is the following characterization of the elliptic genus (cf. 1.6): Whereas the signature may be viewed as the unique genus satisfying

$$\operatorname{sign}(X) = \operatorname{sign}(X^g \circ X^g)$$

for involutions g homotopic to the identity, the elliptic genus is characterized by the property

$$\Phi(X) = \Phi(X^g \circ X^g)$$

for spin manifolds X and involutions g contained in a circle action.

After introducing some material on elliptic genera, the formula

$$\Phi(X)_g = \Phi(X^g \circ X^g)$$

will be derived in Section 1 by means of the Atiyah–Bott–Singer index theorem (Theorem 1.4). We also develop other consequences of the Atiyah–Bott–Singer index theorem which are needed later.

Section 2 deals with homogeneous spin manifolds $X = G/H$. Our main result (Theorem 2.3) follows by induction once we have shown that the self-intersection $X^g \circ X^g$ can itself be realized as a finite union of homogeneous spin manifolds. We also derive a formula (Theorem 2.5) for the signature of a homogeneous space G/H (hence for $\Phi(G/H)$ in the spin case) generalizing an old formula for complex homogeneous spaces [6]. This formula has been used by Bliss *et al.* [7] to compute the signatures of some exceptional symmetric spaces.

In Section 3 we shall have a closer look at specific involutions on homogeneous spaces. These investigations were originally motivated by the search for a formula for the signature and finally led to our Theorem 2.5, now proved by different methods. However, as already indicated by some examples, the methods in Section 3 together with Theorem 1.5 (or [15]) might provide a more effective way to compute $\Phi(X)$ (or sign (X)).

1. Elliptic genera and involutions

1.1. We recall the theory of genera [14] for compact oriented $4k$-dimensional differentiable manifolds. For such a manifold X of dimension $4k$ we write the total Pontrjagin class in the form

$$(1) \qquad p(X) = 1 + p_1 + p_2 + \cdots = \prod_{j=1}^{2k} (1 + x_j^2),$$

where $p_i \in H^{4i}(X, \mathbb{Z})$ and where the x_j are the formal roots considered as 2-dimensional cohomology classes in some extension of the rational cohomology ring of X. Consider a power series

$$Q(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

with coefficients a_i in some commutative algebra A over $\mathbb{C}$. Suppose that Q is even. Then the genus φ_Q of X belonging to the power series Q is defined by

$$(2) \qquad \varphi_Q(X) = \left(\prod_{j=1}^{2k} Q(x_j) \right) [X].$$

We did not request here that $a_0 = 1$. Therefore, it is important to have the formal roots x_j in (1) indexed from 1 to $2k$, where $\dim X = 4k$.

The genus φ_Q is a homomorphism from the bordism ring $\Omega \otimes \mathbb{C}$ to the ring A. It is defined also for non-connected manifolds of mixed dimension not necessarily divisible by 4. If the dimension is not divisible by 4, then the genus is 0.

If the coefficient a_0 of $Q(x)$ is invertible in A, then the power series $a_0^{-1}Q(a_0x)$ defines the same genus. The normalized power series (with constant term equal to 1) are in one-to-one correspondence to all possible genera. For a normalized power series we can use in (1) formal roots x_j indexed from 1 to r provided $r \geqslant k$. The signature of X equals the genus belonging to the power series (see [14, §8])

$$\frac{x}{\operatorname{tgh} x/2} = x\,\frac{1+e^{-x}}{1-e^{-x}} = x\,\frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}} \tag{3}$$

with constant term 2 or equivalently to the normalized power series $x(\operatorname{tgh} x)^{-1}$. The power series in (3) is more natural because it indicates that for a complex manifold the signature equals the holomorphic Euler number with coefficients in the exterior algebra of the dual tangent bundle, or in other words the value of the χ_y-genus for $y = 1$. For spin manifolds Equation (3) indicates that the signature is the index of the Dirac operator twisted with the full spinor bundle (compare [28]).

Let W be a complex vector bundle over X. Then the signature of X with coefficients in W can be defined as

$$\operatorname{sign}(X, W) = \left(\operatorname{ch}(W)\cdot \prod_{j=1}^{n} \frac{x_j}{\operatorname{tgh} x_j/2}\right)[X], \tag{4}$$

where X has dimension $2n$. We use the splitting (1), and $\operatorname{ch}(W)$ denotes the Chern character of W. Indeed, $\operatorname{sign}(X, W)$ is an integer. It equals the index of some elliptic operator, obtained by twisting the signature operator ([4, §6]) with W. For complex manifolds and a holomorphic bundle we have $\operatorname{sign}(X, W) = \chi_1(X, W)$, see [14] and [16]. For odd-dimensional manifolds $\operatorname{sign}(X, W) = 0$.

1.2. We shall define the elliptic genus $\Phi(X)$ using Witten's q-development. For a complex vector bundle W of dimension n

$$\Lambda_t W = \sum_{i=0}^{n} \Lambda^i W\cdot t^i$$

$$S_t W = \sum_{i=0}^{\infty} S^i W\cdot t^i,$$

where $\Lambda^i W$ and $S^i W$ are the exterior and symmetric powers of W. Let T be the complex extension of the tangent bundle of X. The definition of $\Phi(X)$ as a formal power series in q with integral coefficients is

$$\Phi(X) = \operatorname{sign}\left(X, \prod_{n=1}^{\infty} \Lambda_{q^n} T \cdot \prod_{n=1}^{\infty} S_{q^n} T\right). \tag{5}$$

In fact, $\Phi(X)$ is the genus with respect to the power series $Q(x) = x/f(x)$, where

$$f(x) = \frac{1 - e^{-x}}{1 + e^{-x}} \prod_{n=1}^{\infty} \frac{1 - q^n e^{-x}}{1 + q^n e^{-x}} \cdot \frac{1 - q^n e^{x}}{1 + q^n e^{x}}. \tag{6}$$

This follows from (3), (4) and (5) using the formulas for the Chern character of the exterior and symmetric powers of T. For convenience we compare with the notation of [16]: The power series $Q(x)$ corresponds to $\tilde{Q}(x)$ in (15) of [16] for $y = 1$. The genus Φ corresponds to $\tilde{\varphi}_2$ (the level N equals 2). The complex extension T of the tangent bundle of X corresponds to $T \oplus T^*$ in [16]. There T was the complex tangent bundle of a complex manifold.

1.3. Let us now recall the modular properties of the elliptic genus. If we put $q = e^{2\pi i \tau}$ with $\tau \in \mathbb{H}$ (upper half plane), then the infinite product in (6) is compact uniformly convergent in $\mathbb{H} \times \mathbb{C}$, where $\tau \in \mathbb{H}$ and $x \in \mathbb{C}$. The power series $Q(x)$ has constant term $\varepsilon^{-1/4}$, where

$$\varepsilon = \frac{1}{16} \prod_{n=1}^{\infty} \left(\frac{1 - q^n}{1 + q^n}\right)^8 \tag{7}$$

is the well-known modular form of weight 4 for the group $\Gamma_0(2)$ (denoted by ε_S in [29, (19)]). The power series $\varepsilon^{1/4} \cdot Q(x)$ is the normalized power series used for the usual definition of the elliptic genus of a $4k$-dimensional manifold as a modular form of weight $2k$ for $\Gamma_0(2)$. Therefore $\varepsilon^{k/2} \cdot \Phi(X)$ is a modular form of weight $2k$ for $\Gamma_0(2)$, whereas $\Phi(X)$ is a modular function for $\Gamma_0(2)$ (if k is even). In general, $\Phi(X)^2$ is a modular function for $\Gamma_0(2)$. For brevity, we shall call $\Phi(X)$ a modular function for every k.

The modular properties stem from the fact that $f(x)$ as defined in (6) is attached to the lattice $L = 2\pi i(\mathbb{Z}\tau + \mathbb{Z})$. The function $f(x)$ has zeros of order 1 in L and poles of order 1 in $\pi i + L$. The function $f(x)$ satisfies

$$f(x + 2\pi i) = f(x), \quad f(x + 2\pi i \tau) = -f(x).$$

It is elliptic for the lattice $2\pi i(\mathbb{Z} \cdot 2\tau + \mathbb{Z})$. The following property will be very important:

$$f(x + \pi i) \cdot f(x) = 1. \tag{8}$$

The modular function $\Phi(X)$ is holomorphic in $\mathbb{H}$ and also in the cusp given by

$q = 0$ in (5). It assumes the signature of X as value in this cusp. In the other cusp of $\Gamma_0(2)$ the function $\Phi(X)$ has a pole of order $\leqslant k/2$ if $\dim X = 4k$.

In fact, we can transform the other cusp (the local coordinate in this cusp is again denoted by q, not to be confused with the q in (5)) in such a way that the q-development becomes

$$\Phi(X) = q^{-k/2} \cdot \hat{A}\Big(X, \prod_{\substack{n \geqslant 1 \\ n \text{ odd}}} \Lambda_{-q^n} T \cdot \prod_{\substack{n \geqslant 2 \\ n \text{ even}}} S_{q^n} T\Big). \tag{9}$$

Putting $q = e^{2\pi i \tau}$, this is again a modular function for $\Gamma_0(2)$. But as a definition of $\Phi(X)$ we always use (5). Remember that the q in (5) and (9) are local coordinates in different cusps. For development (9) see [20] and [28]. For (6) and (9) compare the formulas (16) and (6) in [29].

Observe that $\hat{A}$-genus with coefficients in a complex vector bundle W (denoted by $\hat{A}(X, W)$) is well defined even if X is not a spin manifold. Formally it equals the Riemann–Roch number $T(X, W)$ in [14, §21, (1*)], if one puts $c_1 = 0$. The coefficients of the q-development in (9) are integral if X is a spin manifold. Then they are indices of the Dirac operator twisted with certain vector bundles [4]. In general, the coefficients are integral except at the prime 2 (see [6, §25]).

1.4. Let G be a compact Lie group operating on the differentiable manifold X by orientation-preserving diffeomorphisms. The coefficient of q^m in (5) is of the form $\operatorname{sign}(X, R_m)$ where R_m is a G-bundle over the G-manifold X, e.g.

$$R_0 = 1, \qquad R_1 = 2T, \qquad R_2 = 2(T + T \otimes T)$$

For $g \in G$, the equivariant signature $\operatorname{sign}(X, R_m)_g$ is defined and therefore also the equivariant elliptic genus

$$\Phi(X)_g = \sum_{m=0}^{\infty} \operatorname{sign}(X, R_m)_g q^m$$

as a power series in q with complex coefficients which, in fact, are algebraic integers. The equivariant genus is defined for manifolds of all dimensions. It vanishes for odd-dimensional manifolds.

According to the fixed point theorem of Atiyah *et al.* [4] we can calculate $\Phi(X)_g$ in terms of the fixed point set X^g of g and the action of g in the normal bundle of the submanifold X^g of X.

We shall study this in the case where g is an (orientation-preserving) involution. The submanifold X^g is not necessarily orientable and, of course, in general not connected. The embedding $X^g \to X$ has an approximation $j: X^g \to X$ which is also an embedding and is transversal to X^g. Then

$j^{-1}(X^g) = X^g \cap j(X^g)$ has as normal bundle in X^g the restriction E of the normal bundle of X^g in X to $X^g \cap j(X^g)$. The normal bundle of $X^g \cap j(X^g)$ in X is isomorphic to $E \oplus E$ and hence canonically oriented because E is even-dimensional. Also $X^g \cap j(X^g)$ is canonically oriented and its oriented cobordism class does not depend on j. We denote it by $X^g \circ X^g$ (self-intersection). In [15] it was pointed out that

$$(10) \qquad \operatorname{sign}(X)_g = \operatorname{sign}(X^g \circ X^g),$$

for this formula Jänich and Ossa [18] gave an elementary proof without using the fixed point formula. The following theorem generalizes (10) to the elliptic genus.

THEOREM. *Let g be a differentiable orientation-preserving involution on X. Then*

$$(11) \qquad \Phi(X)_g = \Phi(X^g \circ X^g).$$

Proof. We can assume that X is even-dimensional. Let $2t = \dim X^g$ (depending on the component of X^g) and $2r = \dim N^g(-1)$ where $N^g(-1)$ is the normal bundle of X^g in X. We use here the notation of [4, p. 582]. Write formally

$$p(X^g) = \prod_{j=1}^{t} (1 + x_j^2)$$

$$p(N^g(-1)) = \prod_{j=1}^{r} (1 + y_j^2).$$

Then

$$(12) \qquad \Phi(X^g \circ X^g) = \left(\prod_{j=1}^{t} \frac{x_j}{f(x_j)} \cdot \prod_{j=1}^{r} \frac{f(y_j)}{y_j} \right) [X^g \circ X^g].$$

This is true though our power series $Q(x) = x/f(x)$ is not normalized. The reason is that $t - r$ is half the dimension of $X^g \circ X^g$. According to the formula of [4, p. 582], we have

$$\operatorname{sign}(X)_g = \{u \cdot e(N^g(-1))\}[X^g]$$

where $e(N^g(-1))$ is the twisted Euler class, u some element of $H^*(X^g)$, and $[X^g]$ the twisted fundamental class of X^g. By the general equivariant index theorem (twisting with R_m and its equivariant Chern character)

$$\operatorname{sign}(X, R_m)_g = \{\operatorname{ch}(R_m \mid X^g)(g) \cdot u \cdot e(N^g(-1))\}[X^g].$$

Standard properties of the twisted Euler class yield

$$\operatorname{sign}(X, R_m)_g = \{\operatorname{ch}(R_m \mid X^g)(g) \cdot u\}[X^g \circ X^g].$$

The class u (see [4, p. 582]) is given by

$$\begin{aligned} u &= \prod_{j=1}^{t} \frac{x_j}{\operatorname{tgh} x_j/2} \cdot \prod_{j=1}^{r} \frac{1}{y_j \cdot \operatorname{tgh}(y_j + \pi i)/2} \\ &= \prod_{j=1}^{t} \frac{x_j}{\operatorname{tgh} x_j/2} \cdot \prod_{j=1}^{r} \frac{\operatorname{tgh} y_j/2}{y_j}. \end{aligned}$$

We have

$$u \cdot \sum_{n=0}^{\infty} \operatorname{ch}(R_n \mid X^g)(g) \cdot q^n = \prod_{j=1}^{t} \frac{x_j}{f(x_j)} \cdot \prod_{j=1}^{r} \frac{1}{y_j \cdot f(y_j + \pi i)}$$

which proves the result in view of (8) and (12).

1.5. If the involution g is homotopic to the identity, then $\operatorname{sign}(X)_g = \operatorname{sign}(X) = \operatorname{sign}(X^g \circ X^g)$. This is not true for the elliptic genus.

EXAMPLE. Let X be the complex projective plane and g a projective involution with X^g consisting of a projective line and a point. Then $X^g \circ X^g$ is a point (with positive orientation) and

$$\Phi(X)_g = 1 \quad \text{(independent of } q\text{)},$$

whereas

$$\Phi(X) = \frac{\delta}{\sqrt{\varepsilon}} = 1 + 32q + \cdots$$

where $\delta = \frac{1}{4} + 6(q + q^2 + 4q^3 + \cdots)$ is a modular form of weight 2 for $\Gamma_0(2)$, denoted by δ_s in [29]. We have, for example, $\operatorname{sign}(X, T) = 16$ and $\operatorname{sign}(X, T)_g = 0$.

If a compact connected Lie group G acts differentiably on the spin manifold X, then $\Phi(X)_g$ does not depend on g for $g \in G$. This is the fundamental rigidity theorem on elliptic genera conjectured by Witten and proved by Taubes and by Bott and Taubes [10]. As a corollary of the rigidity theorem we have

THEOREM. *Let g be an involution contained in the compact connected Lie group G acting differentiably on the spin manifold X, then*

$$(13) \qquad \Phi(X) = \Phi(X^g \circ X^g).$$

The involution g belongs to a circle group contained in G. The circle action is either even or odd (see [3, Lemma 2.4]). (Assume that X is connected.) Therefore, all components of X^g have a codimension $\equiv 0 \bmod 4$ (even action) or a codimension $\equiv 2 \bmod 4$ (odd action). In the odd case all codimensions of $X^g \circ X^g$ are $\equiv 4 \bmod 8$. By (13) and (9) the order of the pole of $\Phi(X)$ for $\dim X = 4k$ is half-integral and integral. Therefore, $\Phi(X) = 0$ which is well known. The vanishing of $\Phi(X)$ means that all coefficients in (9), in particular $\hat{A}(X)$ and $\hat{A}(X, T)$, vanish. Of course, also the coefficients in (5) vanish, in particular sign(X) and sign(X, T). We now consider even actions. By $\operatorname{codim} X^g \geqslant 4r$ we mean that the codimension of each component of X^g is greater than or equal to $4r$.

COROLLARY. *Let g be an involution contained in the compact connected Lie group G acting differentiably on the $4k$-dimensional spin manifold X. If the action is odd, then $\Phi(X) = 0$. Suppose the action is even and* $\operatorname{codim} X^g \geqslant 4r$. *Then $\Phi(X)$ has in the second cusp a pole of order $\leqslant (k/2) - r$. Therefore, in the Laurent series (9) the first r coefficients beginning with $\hat{A}(X)$ vanish. If $r > 0$, then $\hat{A}(X) = 0$. If $r > 1$, then $\hat{A}(X, T) = 0$. If $r > 2$, then $\hat{A}(X, \Lambda^2 T) = 0$. If $r \geqslant k/2$, then $\Phi(X)$ does not depend on q, it equals the signature of X. If $r > k/2$, then $\Phi(X) = 0$.*

The corollary generalizes the theorem on the vanishing of the $\hat{A}$-genus [3].

1.6. The quaternionic projective spaces $P_k(H)$ are $4k$-dimensional spin manifolds. They admit projective involutions with a quaternionic hyperplane $P_{k-1}(H)$ and a point as fixed point set. The theorem in 1.5 (formula (13)) yields

$$\Phi(P_k(H)) = \Phi(P_{k-2}(H))$$

and hence $\Phi(P_k(H)) = 1$ for k even and $=0$ for k odd. Since $P_2(\mathbb{C})$ and the $P_k(H)$ generate the cobordism algebra $\Omega \otimes \mathbb{C}$, the elliptic genus φ is characterized by

$$\Phi(P_2(\mathbb{C})) = \delta/\sqrt{\varepsilon}$$

$$\Phi(P_k(H)) = 1 \quad \text{for } k \text{ even}$$

$$\Phi(P_k(H)) = 0 \quad \text{for } k \text{ odd,}$$

a well-known fact.

For the Cayley plane $\mathbf{W}$ of dimension 16 (cf., e.g., [5, §19]) we get in a similar way

$$\Phi(\mathbf{W}) = 1.$$

1.7. We mention some facts which will be of use later.

(a) Let us consider the situation as in the theorem of 1.5. Then g respects the spin structure of X because the set of possible spin structure is discrete. Then by a theorem of Edmonds used and proved again in [10, Lemma 10.1]), the fixed point set X^g is orientable. The normal bundle of $X^g \circ X^g$ in X (see 1.4) is $E \oplus E$, where E is orientable in our case ($w_1(E) = 0$). Therefore the total Stiefel–Whitney class $w(E \oplus E) = w(E)^2$ equals $1 + w_2(E)^2 +$ higher terms. Hence $E \oplus E$ is a spin bundle. Therefore $X^g \circ X^g$ – or, more precisely, $X^g \cap j(X^g)$ – is a spin manifold, because X is a spin manifold.

(b) Let X be a compact oriented $2n$-dimensional differentiable manifold with a circle action with isolated fixed points. In each fixed point x the tangent space T_xT splits as an S^1-module

$$T_xX = \bigoplus_{i=1}^{n} V(m_i)$$

where $V(m_i)$ is isomorphic as a real S^1-module to $\mathbb{C}$ on which $z \in S^1$ acts by

$$v \mapsto z^{m_i} v \quad (v \in \mathbb{C}).$$

If we choose the rotation numbers $m_i \in \mathbb{Z}$ in such a way that the usual orientations on the summands $V(m_i) \cong \mathbb{C}$ induce the given orientation on T_xX, then they are uniquely defined up to an even number of sign changes. In particular, their product $m_1 \cdot \cdots \cdot m_n \in \mathbb{Z}$ is well defined.

Consider a polynomial $P(p_1, \ldots, p_k)$ of weight k in the universal Pontrjagin classes. For each fixed point x we consider the number

$$P_x = \frac{1}{m_1 m_2 \cdots m_n} \cdot P(\sigma_1, \ldots, \sigma_k)$$

where σ_j is the jth elementary symmetric function in the m_i^2. If we now replace the universal p_j by the Pontrjagin classes of X, then we have for $n = 2k$

$$P(p_1 \cdots p_k)[X] = \sum_{x \in X^{S^1}} P_x. \tag{14}$$

This formula and similar formulas for Chern numbers are due to Bott [9] and can be deduced from the fixed point formula of Atiyah–Bott–Singer: Every Pontrjagin number $P(p_1, \ldots, p_k)[X]$ is a linear combination of numbers $\operatorname{sign}(X, W)$ where W is associated to the tangent bundle of X by some complex representation. The finite Laurent series $\operatorname{sign}(X, W)_\lambda$ (for a general element λ of S^1) can be calculated by the fixed point theorem for isolated fixed points. Then specializing to $\lambda = 1$ gives (14).

If the action has no fixed points (though it is not necessarily free) it follows that all Pontrjagin numbers and hence all genera of X are 0. Then the manifold X represents the 0 of the bordism ring $\Omega \otimes \mathbb{C}$.

REMARK. If $n \neq 2k$, we can still ask for a meaning of the right-hand side of (14). The answer (which can also be obtained from the fixed point formula) is the following. Consider the universal S^1-bundle E over the infinite-dimensional complex projective space with $g \in H^2(P_\infty(\mathbb{C}), \mathbb{Z})$ being the standard generator. Take the associated bundle E_X with X as fibre using the S^1-action on X. Then $P(p_1, \ldots, p_k)$ taken for the bundle along the fibres of E_X defines a $4k$-dimensional cohomology class of E_X which can be integrated over the fibre to give a $(4k - 2n)$-dimensional class of $P_\infty(\mathbb{C})$ which equals

$$\left(\sum_{x \in X^{S^1}} P_x\right) \cdot g^{2k-n}.$$

Compare [23]. In particular (see [9]),

$$\sum_{x \in S^{S^1}} P_x = 0 \quad \text{for } 2k < n.$$

(c) The signature has the following very special property: The equivariant signature does not depend on λ. The rigidity is an immediate consequence of the topological definition of the signature and implies that for the L-polynomials

$$\sum_{x \in X^{S^1}} (L_k)_x = 0 \quad \text{for } 4k \neq \dim X.$$

This corresponds to the strict multiplicativity of the L-polynomials ([6, §28.4]). The rigidity of the signature is also a consequence of the fixed point theorem: The contribution for $\operatorname{sign}(X)_\lambda$ coming from a fixed point equals

$$(15) \qquad \prod_{i=1}^{n} \frac{1 + \lambda^{-m_i}}{1 - \lambda^{-m_i}}.$$

The basic idea is to consider $\operatorname{sign}(X)_\lambda$ as a rational function in λ (compare [3]). As a finite Laurent series it can have poles only in 0 and ∞ whereas the contribution coming from the fixed points have poles only in roots of unity. Hence $\operatorname{sign}(X)_\lambda$ is a constant whose value can be obtained for $\lambda \to \infty$ in (14). The limit of (15) for the rotation numbers m_i in the fixed point x equals

$$(-1)^{\mu(x)}$$

where

$$\mu(x) = \#\{i \mid m_i < 0\}.$$

Note that the parity of $\mu(x)$ is well defined. Thus the following formula results.

It was mentioned in [3, p. 26] and earlier in [4, p. 594.]:

$$\text{(16)} \qquad \operatorname{sign}(X) = \sum_{x \in X^{S^1}} (-1)^{\mu(x)}.$$

2. Homogeneous spaces

Let X be a compact spin manifold homogeneous under the action of a compact Lie group G. The aim of this chapter is to show that the elliptic genus $\Phi(X)$ is a constant modular function. If one defines the elliptic genus as a modular form (see 1.3), then this elliptic genus of X equals a constant times a power of ε.

2.1. Let X be an arbitrary homogeneous space of the compact Lie group G, and let $\Gamma \subset G$ be an arbitrary subgroup (not necessarily Lie). Then the centralizer

$$M = C_G(\Gamma) = \{g \in G \mid g\gamma = \gamma g \text{ for all } \gamma \in \Gamma\}$$

is a closed subgroup of G, thus a compact Lie subgroup (see, e.g., [2, 2.26, 2.27]). Similarly, the fixed point set $Y = X^\Gamma = \{x \in X \mid \gamma x = x \text{ for all } \gamma \in \Gamma\}$ is a closed submanifold of X (without loss of generality we can replace Γ by its closure $\bar{\Gamma}$ in G which is a compact Lie subgroup, see above). It is clear that M acts on Y.

PROPOSITION. *The manifold Y decomposes into a finite union of M-orbits.*

Proof. Since M and Y are compact, it is sufficient to show for any point $y \in Y$ that the dimension of the M-orbit $Z = M \cdot y$ of y equals the dimension of Y in y or that the tangent spaces $T_y Z$ and $T_y Y$ coincide. Consider the map

$$\varphi \colon G \to X, \ \varphi(g) = g \cdot y$$

and its differential in the unit element e of G:

$$D_e\varphi \colon \mathfrak{g} \to T_y X$$

(here $\mathfrak{g} = \operatorname{Lie} G = T_e G$). We let Γ act on G by conjugation and on $\mathfrak{g}$ by the induced adjoint action. With respect to the natural action of Γ on X and $T_y X$ the map φ and hence $D_e\varphi$ become Γ-equivariant. Since we can replace Γ by its compact closure $\bar{\Gamma}$ we get splittings of Γ-modules

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{n}, \qquad T_y X = T_y Y \oplus N_y Y$$

where $\mathfrak{m} = \mathfrak{g}^\Gamma = \operatorname{Lie} K$ and $T_y Y = (T_y X)^\Gamma$. By Γ-equivariance we get

$$D_e\varphi(\mathfrak{m}) \subset T_y Y \quad \text{and} \quad D_e\varphi(\mathfrak{n}) \subset N_y Y.$$

Since X is homogeneous, $D_e\varphi$ is surjective. Thus $T_yZ = D_e\varphi(\mathfrak{m}) = T_yY$ which had to be shown.

REMARK. The proof of the proposition above goes through for algebraic groups G over algebraically closed fields of characteristic zero provided one assumes Γ, or its Zariski closure $\bar{\Gamma}$ in G, to be reductive. (Of course, X is assumed to be algebraic, too, i.e. $H \subset G$ has to be Zariski closed.)

2.2. The following transversality result will be useful in the next section.

PROPOSITION. *Let G be a Lie group, X a homogeneous space of G, and $Y \subset X$ a submanifold. Then there exist $g \in G$ such that Y and its translate gY intersect transversally.*

Proof. The map $\Psi: G \times Y \to X$, $\Psi(g, y) = gy$, is a submersion. Thus

$$Z = \Psi^{-1}(Y) = \{(g, y) \in G \times Y \mid gy \in Y\}$$

is a submanifold of $G \times Y$. Let $\pi: Z \to G$, $\pi(g, y) = g$, denote the first projection and let $g \in G$ be a regular value of π which exists by Sard's theorem. The regularity of g is equivalent to the transversality of Z and $\{g\} \times Y$ in $G \times Y$. Since Ψ is submersive, this implies the transversality of the submanifolds

$$Y = \Psi(Z) \quad \text{and} \quad gY = \Psi(\{g\} \times Y) \quad \text{in } X.$$

REMARKS. (1) If $Y \subset X$ is equidimensional of codimension k, then $Y \cap gY = \Psi(\pi^{-1}(g))$ is a submanifold of codimension $2k$ in X (or empty).

(2) By Sard's theorem, the set of $g \in G$ with Y and gY transversal is dense in G. If G, X and Y are algebraic, it is also Zariski-open. For an elaboration of that situation cf. [19], from which we have taken the basic idea of the proof (cf. also [12, II, §4, Remark after Lemma 4.6]).

2.3. We can now establish our main result.

THEOREM. *Let X be a connected homogeneous space of a compact Lie group G. Assume that X is oriented and admits a spin structure. Then the elliptic genus $\Phi(X)$ is a constant modular function*

$$\Phi(X) = \operatorname{sign}(X). \tag{17}$$

If $\dim X \not\equiv 0 \bmod 8$, *then* $\Phi(X) = \operatorname{sign}(X) = 0$.

Proof. If $\Phi(X)$ is a constant, then it equals $\operatorname{sign}(X)$ by 1.3. We shall proceed by induction on the dimension of X, the case $\dim X = 0$ being trivial. We may assume that G is connected and that it acts faithfully on X. Let $\gamma \in G$ be a non-trivial involution in G and $Y = X^\gamma$ its fixed point set. According to Theorem 1.5 we have

$$\Phi(X) = \Phi(Y \circ Y).$$

By Proposition 2.2 we can realize $Y \circ Y$ as the transversal intersection of Y with a generic translate gY, $g \in G$. The intersection $Y \cap gY$ equals X^{Γ}, where Γ is the subgroup of G generated by γ and $g\gamma g^{-1}$. Thus, by Proposition 2.1, the intersection $Y \cap gY$ is a finite union of homogeneous spaces under the compact group $M = C_G(\Gamma)$ (or its identity component M^0). According to 1.4 and 1.7(a), the connected components of $Y \cap gY$ are canonically oriented and spin manifolds. We conclude the proof of (17) by applying the induction hypothesis to these components. If γ is an odd involution, then $\Phi(X) = 0$ (see 1.5). If γ is even, then $\dim X \equiv \dim(Y \cap gY) \bmod 8$, possibly $Y \cap gY$ is empty, and induction can also be used to prove $\Phi(X) = 0$ for $\dim X \not\equiv 0 \bmod 8$.

2.4. Many homogeneous manifolds bound modulo torsion, i.e. they represent the 0 of the bordism algebra $\Omega \otimes \mathbb{C}$. All genera vanish on these manifolds.

LEMMA. *Let $X = G/H$ be a connected homogeneous space under the compact Lie group G. Assume* $\operatorname{rank}(H) < \operatorname{rank}(G)$. *Then X admits an S^1-action with empty fixed point set, $X^{S^1} = \emptyset$.*

Proof. Let $T \subset H$ be a maximal torus of H contained in a maximal torus $S \subset G$ of G and let $W = N_G(S)/S$ be the Weyl group of S in G. Let $X(T) = \operatorname{Hom}(S^1, T) \subset \operatorname{Lie}(T)$ and $X(S) = \operatorname{Hom}(S^1, S) \subset \operatorname{Lie}(S)$ denote the lattices of co-characters. Since $\operatorname{rank}_{\mathbb{Z}}(X(T)) = \operatorname{rank}(H) < \operatorname{rank}(G) = \operatorname{rank}_{\mathbb{Z}}(X(S))$ we can find a homomorphism $\lambda\colon S^1 \to S$ corresponding to a point in the complement of the finite union

$$\bigcup_{w \in W} w \cdot X(T)$$

(note that W acts on $X(S)$ by conjugation). We claim that the S^1-action on X induced by λ has no fixed point on X. Otherwise $\lambda(S^1)$ would be conjugate into H, thus into T. However, elements of $X(S)$ are G-conjugate if and only if they are W-conjugate. Thus $\lambda \in \bigcup_{w \in W} w \cdot X(T)$ contradicting our choice of λ. (For the conjugacy results used, cf. [2, 4.21 and 4.33].)

REMARKS. (1) The result above is due to Hopf and Samelson [17, §6].

(2) Note that the S^1-action constructed above need not be free. For example, all involutions of $\mathrm{SU}(n)$ are conjugate into the subgroup $\mathrm{SO}(n)$. Thus any involution of $\mathrm{SU}(n)$ has a fixed point on $X = \mathrm{SU}(n)/\mathrm{SO}(n)$.

Let $G \supset H$ be compact connected Lie groups with $\operatorname{rank}(G) = \operatorname{rank}(H)$ and a common maximal torus $T \subset H$. Let $\Sigma(T, H)$ and $\Sigma(T, G)$ denote the corresponding root systems in $X^*(T) = \operatorname{Hom}(T, S^1)$. We call H *small* in G if there exists a root $\alpha \in \Sigma(T, G)$ which is orthogonal to all roots in $\Sigma(T, H)$.

PROPOSITION. *Let $X = G/H$ be a connected homogeneous space under the compact Lie group G and assume that either* $\operatorname{rank}(H) < \operatorname{rank}(G)$ *or that H is small in G. Then X bounds modulo torsion, and all genera, in particular the elliptic genus $\Phi(X)$, vanish.*

Proof. If $\operatorname{rank}(H) < \operatorname{rank}(G)$ combine the above lemma with 1.7(b). If H is small in G let $\alpha \in \Sigma(T, G)$ be a root orthogonal to $\Sigma(T, H)$ and let $G_\alpha \subset G$ denote the rank-1 subgroup of G corresponding to α ($G_\alpha \cong \mathrm{SU}(2)$ or $\mathrm{SO}(3)$). Then G_α and H commute, and their product forms a closed subgroup H' of G with $\operatorname{rank}(H') = \operatorname{rank}(G)$. The natural projection $G/H \to G/H'$ realizes X as a G-bundle over G/H' with fibre $H'/H \cong G_\alpha/G_\alpha \cap T \cong S^2$. Thus X bounds the associated G-bundle over G/H' with fibre the 3-disk D^3.

REMARKS. (1) The proposition shows that our theorem in 2.3 is interesting only for $X = G/H$ where G and H have equal rank and where H is large (i.e. not small) in G.

(2) A necessary condition for H to be small in G is that the rank of the root system $\Sigma(T, H)$ is strictly smaller than the rank of $\Sigma(T, G)$. This condition is by no means sufficient (for example, $\mathrm{U}(2) \times \mathrm{U}(2) \subset \mathrm{U}(4)$ and, more generally, $\mathrm{U}(2)^n \subset \mathrm{U}(2n)$ are not small).

2.5. As another application of the Atiyah–Bott–Singer fixed point theorem we shall now derive a formula for the signature of a homogeneous space $X = G/H$ under a compact connected Lie group G. By our results in 2.4 we may concentrate on the case $\operatorname{rank}(G) = \operatorname{rank}(H)$. We may also assume that X is simply connected which, in this case, is equivalent to H being connected.

Let $T \subset H \subset G$ be a common maximal torus and $X^*(T) = \operatorname{Hom}(T, S^1)$ its character lattice which is in W-equivariant duality to the lattice $X(T) = \operatorname{Hom}(S^1, T)$ of co-characters

$$X^*(T) \times X(T) \to \mathbb{Z}$$

$$(\alpha, \lambda) \to \langle \alpha, \lambda \rangle$$

defined by

$$\alpha(\lambda(z)) = z^{\langle \alpha, \lambda \rangle}$$

for all $\alpha \in X^*(T)$, $\lambda \in X(T)$, $z \in S^1$.

The root system $\Sigma' = \Sigma(T, H)$ of H is contained in the root system $\Sigma = \Sigma(T, G) \subset X^*(T)$ of G. We let $W' = W(H) = N_H(T)/T$ and $W = W(G) = N_G(T)/T$ denote the corresponding Weyl groups. Let $\Sigma^+ \subset \Sigma$ be a system of positive roots with basis $\Delta \subset \Sigma^+$ of simple roots. Then $\Sigma'^+ = \Sigma' \cap \Sigma^+$ is a system of positive roots in Σ'. Let $\Delta' \subset \Sigma'^+$ be the basis of simple roots (we need not have $\Delta' \subset \Delta$!). Let $\Psi = \{\alpha \in \Sigma^+ \mid \alpha \notin \Sigma'\}$ denote the set of complementary positive roots. For any subset $\Omega \subset \Sigma^+$ and any $w \in W$ we put

$$\Omega(w) = \{\alpha \in \Omega \mid w^{-1}(\alpha) \notin \Sigma^+\}.$$

The following result may in principle be deduced from the orientability of X. However, it may be quite satisfactory to have a purely root-theoretic proof.

PROPOSITION 1. *Let $w \in W$ and $\mu(w) = \operatorname{card} \Psi(w)$. Then $\mu(vw) \equiv \mu(w) \bmod 2$ for all $v \in W'$.*

Proof. Let $l = l_\Delta \colon W \to \mathbb{N}$ and $l' = l_{\Delta'} \colon W' \to \mathbb{N}$ denote the length functions on the Weyl groups corresponding to the systems of simple roots Δ and Δ'. For any $w \in W$ and $v \in W'$ we have (cf. [11, Cor. 2, p. 158])

$$l(w) = \operatorname{card} \Sigma^+(w) \quad \text{and} \quad l'(v) = \operatorname{card} \Sigma'^+(v).$$

Thus, for all $v \in W'$

$$\mu(v) = l(v) - l'(v).$$

Since any reflection in W has determinant -1 (on $X^*(T) \subset (\operatorname{Lie} T)^*$) we have

$$(-1)^{l(v)} = \det(v) = (-1)^{l'(v)}$$

and hence $\mu(v) \equiv 0 \pmod 2$ for all $v \in W'$. From this we can deduce the general case of our assertion. Let $v \in W'$. We decompose

$$\Psi = \Psi(v) \cup \Psi(v)^0$$
$$-\Psi = -\Psi(v) \cup -\Psi(v)^0$$

where $\Psi(v)^0 = \{\alpha \in \Psi \mid v^{-1}(\alpha) \in \Sigma^+\}$. Since $\Psi \cup -\Psi = \Sigma \backslash \Sigma'$ is stable under W' we get

$$(*) \qquad v(\pm\Psi(v)) = \mp\Psi(v)$$
$$v(\pm\Psi(v)^0) = \pm\Psi(v)^0.$$

Now let $w \in W$. We consider the induced decompositions

$$\Psi(w) = (\Psi(w) \cap \Psi(v)) \cup (\Psi(w) \cap \Psi(v)^0)$$
$$\Psi(vw) = (\Psi(vw) \cap \Psi(v)) \cup (\Psi(vw) \cap \Psi(v)^0).$$

From $(*)$ it is clear that

$$\operatorname{card}(\Psi(vw) \cap \Psi(v)^0) = \operatorname{card}(\Psi(w) \cap \Psi(v)^0)$$

and

$$\begin{aligned}\operatorname{card}(\Psi(vw) \cap \Psi(v)) &= \operatorname{card}\{\beta \in -\Psi(v) \mid w^{-1}(\beta) \in -\Sigma^+\} \\ &= \operatorname{card}\{\alpha \in \Psi(v) \mid w^{-1}(\alpha) \in \Sigma^+\} \\ &= \operatorname{card} \Psi(v) - \operatorname{card}(\Psi(w) \cap \Psi(v)).\end{aligned}$$

Since card $\Psi(v) = \mu(v)$ is even we finally get

$$\text{card } \Psi(vw) \equiv \text{card } \Psi(w) \pmod 2.$$

REMARK. In the case where Σ' is rationally closed in Σ we get $\Delta' \subset \Delta$. Then Ψ is stable under W' and we even have $\mu(vw) = \mu(w)$ for all $w \in W$, $v \in W'$. Examples (e.g. $A_1 \times A_1 \subset B_2$) show that, in general, Ψ need not be stable under W'.

The following result is well known. Let $\lambda \in X(T) = \mathrm{Hom}(S^1, T)$ be a regular one-parameter subgroup into G and denote the fixed point set of $\lambda(S^1)$ on $X = G/H$ by X^λ.

PROPOSITION 2. *We have*

$$X^\lambda = X^T = \{wH \in G/H \mid w \in W\}.$$

In particular, X^λ is a finite set in bijection to $W(G)/W(H)$.

Proof. (compare [17, §§5, 7]). The inclusions '$\supset$' are obvious. Conversely, let $x = gH \in X^\lambda$. Then $\lambda(S^1) \subset gHg^{-1}$. Let $T' \subset gHg^{-1}$ be a maximal torus containing $\lambda(S^1)$. Since T' is a maximal torus of G, too, and since $\lambda(S^1)$ is contained in a unique maximal torus of G (see, e.g., [2, 4.35, 4.41]) we must have $T = T' \subset gHg^{-1}$ or $g^{-1}Tg \subset H$. Let $h \in H$ be such that $h^{-1}g^{-1}Tgh = T \subset H$ (use [2, 4.23]), i.e. such that $gh \in N_G(T)$. If we denote by w the image of gh in w we have (with the usual abuse of language)

$$x = gH = ghH = wH$$

which had to be shown.

With the notations introduced above we can now prove:

THEOREM. *Let $X = G/H$ be a simply connected homogeneous space under the compact Lie group G and assume* $\mathrm{rank}(H) = \mathrm{rank}(G)$. *Then*

(i) $e(X) = \#(W(G)/W(H))$

(ii) $\pm\mathrm{sign}(X) = (1/\# W(H)) \sum_{w \in W} (-1)^{\mu(w)}$.

Proof. (i) is a classical result due to Hopf and Samelson (see Remark 1 below). To prove (ii) we first fix an orientation on X (for which the positive sign will result). It is sufficient to fix an orientation on the tangent space $T_H X \cong \mathfrak{g}/\mathfrak{h}$ which decomposes under the action of T into a direct sum

$$T_H X = \bigoplus_{\alpha \in \Psi} V_\alpha,$$

where the summand V_α may be identified as an $\mathbb{R} - T$ module with $\mathbb{C}$ on which T acts by the root α

$$tv = \alpha(t) \cdot v \quad (t \in T,\ v \in \mathbb{C}).$$

Via this identification each V_α and thus $T_H X$ obtain an orientation. Now choose a regular $\lambda \in X(T)$ inside the fundamental chamber. Then the rotation numbers of $\lambda(S^1)$ in the fixed point $H \in X = G/H$ are given by the strictly positive numbers

$$\langle \alpha, \lambda \rangle, \quad \alpha \in \Psi.$$

Computing the rotation numbers in another fixed point $w^{-1}H \in X$ is equivalent to computing the rotation numbers of $w(\lambda)$ in $H \in X$, which are

$$\langle \alpha, w(\lambda) \rangle = \langle w^{-1}(\alpha), \lambda \rangle, \quad \alpha \in \Psi.$$

Thus there are $\mu(w)$ negative rotation numbers of λ in $w^{-1}H$ and we obtain (ii) from Proposition 2 above and 1.7(c), (16).

REMARKS. (1) We have also mentioned (i) since, in the context of our proof of (ii), it can be deduced from the classical Lefschetz fixed point formula (for more details compare [17], see also [2, proof of 4.21], where G/T is treated).

(2) Note that G/H carries no canonical orientation. Therefore we have admitted both signs of sign(X). If G/H admits a homogeneous complex structure, one can fix an orientation by fixing such a structure.

(3) In this last case, our formula for sign(X) is already proved in [6, Th. 24.3], where the Hodge-theoretic expression for the signature gives:

$$\operatorname{sign}(X) = \sum_{p=0}^{n} (-1)^p b_{2p}$$

($n = \dim_{\mathbb{C}} X$, $b_{2p} = 2p$th Betti number). This formula is no longer valid in our general context (e.g. consider $X = S^4 = \mathrm{SO}(5)/\mathrm{SO}(4)$). In the complex case Σ' is rationally closed in Σ. Thus Ψ is W'-stable and the numbers $\mu(w)$ (not only $(-1)^{\mu(w)}$) are invariants attached to the fixed points $w^{-1}H$, $w \in W$. They can be used to express the Poincaré series of the cohomology ring $H^*(G/H, \mathbb{Q})$ (cf. *loc. cit.*)

$$\sum_{p=0}^{n} b_{2p} t^p = \frac{1}{\# W(H)} \sum_{w \in W} t^{\mu(w)}.$$

This formula, too, breaks down in the general, non-complex case (e.g. the right-hand side gives $\frac{1}{2}(t^2 + 2t + 1)$ for the above example $X = S^4$).

(4) Our use of the function $\mu(w)$ related to $w^{-1}H$ instead of $\mu'(w) = \mu(w^{-1})$ related to wH follows tradition.

(5) Our formula for sign(X) remains valid for not necessarily simply connected spaces $X = G/H$ as long as X is orientable. In this case $W(H)$ need not be a Weyl group of a root subsystem. Note that we have made no use of

Proposition 1 in our proof of (ii). Instead, Proposition 1 is a corollary of the orientability of X.

(6) To evaluate $\operatorname{sign}(X)$ requires a summation over the $W(H)$-right cosets of $W(G)$ only. Nonetheless, this may not be easily manageable in complicated cases (e.g. $G = E_8$). In our table at the end, we list the (positive) value of $\operatorname{sign}(X)$, X a symmetric space, provided its value has been computed.

2.6. The rigidity theorem [10] and our Theorem 1.5 hinge on the existence of a spin structure on the manifold X. Here we specify the well-known existence criteria for such a structure in the situation where X is a homogeneous space of the form G/H with $\operatorname{rank}(G) = \operatorname{rank}(H)$.

We fix a common maximal torus $T \subset H \subset G$, root systems

$$\Sigma' = \Sigma(T, H) \subset \Sigma = \Sigma(T, G) \subset X^*(T)$$

and systems of positive roots Σ^+ and $\Sigma'^+ = \Sigma' \cap \Sigma^+$. Let $\Psi = \Sigma^+ \backslash \Sigma'^+$ denote the complementary positive roots. To any subset $\Omega \subset \Sigma^+$ we attach a 'spin weight'

$$\rho_\Omega := \frac{1}{2} \sum_{\alpha \in \Omega} \alpha,$$

which is an element of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

PROPOSITION. *Set G be a connected, simply connected, compact Lie group, H a closed, connected subgroup with* $\operatorname{rank}(H) = \operatorname{rank}(G)$, *and $X = G/H$. Then the following conditions are equivalent*:

(i) *X admits a spin structure;*
(ii) *X admits a unique spin structure;*
(iii) *the second Stiefel–Whitney class $w_2(X)$ vanishes;*
(iv) *the tangential representation $\tau\colon H \to \mathrm{SO}(T_H X)$ lifts to the corresponding spin group*:

$$\begin{tikzcd} & \mathrm{Spin}(T_H X) \arrow[d, "\text{nat.}"] \\ H \arrow[ur, "\sigma"] \arrow[r, "\tau"] & \mathrm{SO}(T_H X). \end{tikzcd}$$

(v) $\rho_\Psi \in X^*(T)$;
(vi) $\rho_{\Sigma'^+} \in X^*(T)$.

Proof. For the equivalence of (i), (ii), (iii) see [6, p. 350] and [21] (note that X is simply connected under our assumptions). It is obvious that (iv) implies (i). Conversely, let $P \to X$ denote the principal $\mathrm{Spin}(T_H X)$ bundle 'restricting' the principal $\mathrm{SO}(T_H X)$ bundle $Q = G \times^H \mathrm{SO}(T_H X) \to X$ of some G-invariant

Riemannian metric on X. Since G is simply connected we get compatible G-actions on the triangle

$$\begin{tikzcd} P \arrow[d] \arrow[dr] & \\ Q \arrow[r] & X \end{tikzcd}$$

(use [3, Prop. 2.1]). Thus P is associated to the H-principal bundle $G \to G/H$ and a lift $\sigma: H \to \mathrm{Spin}(T_H X)$ of $\tau: H \to \mathrm{SO}(T_H X)$. Condition (v) is a well known reformulation of (iv) in terms of the weights of τ. Finally, (v) and (vi) are equivalent since

$$\rho_{\Sigma^+} = \rho_{\Sigma'^+} + \rho_\Psi$$

is an element of $X^*(T)$ (again since G is simply connected).

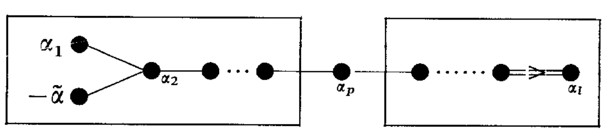
A number of cases can be dealt with quite easily.

COROLLARY. *Let G and H be as in the proposition. Assume that H contains no simple factor of type A_l (l odd), B_l, C_l ($l \equiv 1, 2(4)$), D_l ($l \equiv 2, 3(4)$), or E_7. Then $X = G/H$ admits a spin structure.*

Proof. A glance at the tables of Bourbaki [11, Planches, entry (VII)] shows that $\rho_{\Sigma'^+} \in \mathbb{Z}\Sigma' \subset X^*(T)$ under these conditions (partly, one might also invoke that the order of the fundamental group of an adjoint group of type A_{2l}, E_6, E_8, F_4, G_2 is odd).

In the tables at the end of our article we have indicated by a '+' (resp. '−') the existence (resp. non-existence) of a spin structure on symmetric spaces of the form G/H, $\mathrm{rank}(H) = \mathrm{rank}(G)$.

To illustrate the use of condition (iv) let us deal with the symmetric space $X = B_l/B_q \times D_p$, $p + q = l$, or $X = \mathrm{SO}(2l+1)/\mathrm{SO}(2q+1) \times \mathrm{SO}(2p)$. We can choose bases of simple roots according to the following extended Dynkin diagram (notations of [11]):



D_p B_q

The weight lattice $X^*(T) = P(B_l)$ of $\mathrm{Spin}(2l+1)$ is generated by the roots $\alpha_1, \ldots, \alpha_l$ and the spin weight $\omega_l = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + l\alpha_l)$. Let $Q(B_l)$ denote the root lattice, generated by the α_i only. Then we have (in obvious notation)

$$\rho_{D_p} \equiv \frac{p(p-1)}{2}(\alpha_1 - \tilde{\alpha}) \equiv 0 \bmod Q(B_l)$$

$$\rho_{B_q} \equiv \tfrac{1}{2}(\alpha_{p+1} + \alpha_{p+3} + \cdots + \alpha_{l(-1)}) \bmod Q(B_l).$$

Thus

$$\rho_{\Sigma'^+} = \rho_{D_p} + \rho_{B_q} \equiv \tfrac{1}{2}(\alpha_{p+1} + \cdots + \alpha_{l(-1)}) \bmod Q(B_l).$$

This element lies in $X^*(T) = P(B_l)$ exactly when $p = 0$ (which we need not consider) or when $p = l$, i.e. $q = 0$. Hence

$$X = \mathrm{SO}(2l+1)/\mathrm{SO}(2q+1) \times \mathrm{SO}(2p)$$

admits no spin structure for $q > 0$, whereas

$$X = \mathrm{SO}(2l+1)/\mathrm{SO}(2l) = S^{2l}$$

admits a spin structure (which is of course also clear by condition (iii)).

In the other cases of our table one can either use the corollary or a similar reasoning. In addition, note that for hermitian symmetric spaces one may also employ the computation of the first Chern class $c_1(X) \in H^2(X, \mathbb{Z})$ in [5, 16.1] since $w_2(X)$ is the image of $c_1(X)$ under the natural map $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}/2\mathbb{Z})$.

3. Involutions on homogeneous spaces

We shall now have a closer look at involutions on a homogeneous space X and shall show how formula (10) in 1.4, resp. Theorem 1.5, leads, in some cases at least, to an effective determination of the signature, resp. the elliptic genus $\Phi(X)$.

3.1. As previously, we may restrict our attention to spaces of the form $X = G/H$ where G and H are compact of the same rank. Then the centre of G is contained in H and we may choose the global structure of G according to convenience.

Let us now assume that G is of adjoint type (thus semisimple). We shall recall some facts about the involutions of G and the associated symmetric spaces. Let $\gamma \in G$ be a non-trivial involution and let $K = C_G(\gamma)$ be its centralizer in G. Then $S = G/K$ is a compact symmetric space with simply connected covering $\tilde{S} = G/K^0$ (cf. [13, Ch. VII]). Let $g\gamma g^{-1}$ and gKg^{-1} be generic conjugates of γ and K under G. Then the intersection

$$M = K \cap gKg^{-1} = C_G(\Gamma)$$

(where Γ is the subgroup of G generated by γ and $g\gamma g^{-1}$) can be regarded as a generic isotropy group (i.e. a principal isotropy group) for the left-action of K on $S = G/K$. According to the theory (cf. [13, VII, §§3, 8]) this group can be described as the centralizer in K

$$(18) \qquad M = C_K(A) = K \cap C_G(A)$$

of a maximal γ-split torus $A \subset G$. Here a torus $A \subset G$ is called γ-split if

$$\gamma a \gamma^{-1} = a^{-1} \quad \text{for all } a \in A$$

(this notation follows [25] and is motivated by the relative theory of reductive groups over arbitrary fields; cf., e.g., [26], [27], [24]). All maximal γ-split tori in G are conjugate by an element of K^0 (cf. [13, Ch. V, Lemma 6.3]) and M plays an important part in the classification theory of involutions (cf. [25]). In Table I in Section 4 we have listed all conjugacy classes of involutions in simple groups together with associated invariant objects and numbers:

The *affine coordinate diagram* allows us to specify a representative γ of the conjugacy class in the following way. Let $T \subset G$ be a maximal torus and let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be a system of simple roots. Then γ is determined as an element of T by the conditions

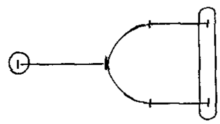
$$(19) \qquad \alpha_i(\gamma) = (-1)^{m_i}, \quad i = 1, \ldots, l,$$

where m_i is the coefficient attached to the simple root α_i in the diagram. (If $-\tilde{\alpha}$ denotes the negative of the highest root, then we shall automatically have

$$(-\tilde{\alpha})(\gamma) = (-1)^{\tilde{m}},$$

where $\tilde{m}$ is the coefficient of the extra node $-\tilde{\alpha}$ in the affine diagram). The affine coordinate diagram was introduced by V. Kac in the more general context of classifying elements of finite order in G (cf. [13, Ch. X, §5]). It also determines the infinitesimal structure, i.e. the Lie algebra $\mathfrak{k}$ of $K = C_G(\gamma)$ whose Dynkin diagram is provided by the subdiagram formed by the vertices with 0 coefficient. Of course, $\dim S = \dim G - \dim K$ is known then.

The *index diagram* attached to γ can be derived as in [25] (where it is called the Araki diagram) or in the following way. Let $\check{G}$ denote the real algebraic group (of adjoint type) giving rise to the non-compact symmetric space $\check{S}$ dual to S (see, for example, [13, Ch. V, §§2, 5]; note that $\check{G}$ need not be connected as a real Lie group). Then the index diagram is the index in the sense of Tits (cf. [26], [27], [24]) attached to $\check{G}$. The dimension of a maximal γ-split torus A, or the rank of S and $\check{S}$, is then given by the number of circles in the index diagram, i.e. two in the example



The Dynkin diagram of the semisimple part of $C_G(A)$, or $M = C_K(A)$ (or $C_{\check{G}}(\check{A})$, where $\check{A}$ is a maximal $\mathbb{R}$-split torus of $\check{G}$ corresponding to A) is the subdiagram formed by the uncircled vertices. The Lie algebra $\mathfrak{m}$ of M is obtained from $\mathfrak{c} = \operatorname{Lie} C_G(A)$ by splitting off the Lie algebra $\mathfrak{a}$ of A. Global information on M can be deduced from the following result:

PROPOSITION. *The following properties hold*:

(i) *$C_G(A)$ is connected.*
(ii) *The commutator subgroup of $C_G(A)$ is contained in K.*
(iii) *The centre of $C_G(A)$ is a γ-stable torus of the form*

$$(S^1 \times S^1)^a \times (S^1)^b,$$

where $a + b = \dim A$ and where γ acts (by conjugation) as follows: on a factor $S^1 \times S^1$ by $(s, t) \mapsto (t, s)$, on a factor S^1 by $t \mapsto t^{-1}$.

Proof. (i) follows since $C_G(A)$ is the centralizer of a torus (cf. [13, Ch. VII, 2.8]), (ii) and (iii) follow from analogous statements for $C_{\check{G}}(\check{A})$ in [24, II, 2.2 and 4.1] which are easily transferred to our situation.

We remark that also the relative root system and the Weyl group $N_K(A)/C_K(A)$ of S can be derived from the index (cf. [27], [13, Ch. X, Table VI]).

3.2. We keep the general assumptions and notations from 3.1. Since the self-intersection $X^\gamma \circ X^\gamma$ relevant in Theorem 1.5 can be realized in the form X^Γ (Γ as in 3.1) and since X^Γ consists of a finite union of orbits under $M = C_G(\Gamma)$ (Proposition 2.1) it is natural to look for involutions γ with M as small as possible.

An involution $\gamma \in G$ is called *quasi-split* if the centralizer $C_G(A)$ of a maximal γ-split torus $A \subset G$ is a maximal torus of G. It is called *split* if there is a maximal torus of G which is γ-split.

REMARK. These definitions are inspired by those of [27]. It is easily seen that they are equivalent to those of [25] when transferred to the compact case. Note that any split involution is quasi-split.

The classification of involutions shows that any simple adjoint group G contains exactly one conjugacy class of quasi-split involutions (characterized by an index diagram all of whose vertices are encircled, cf. also [25]). These involutions are split exactly when the 'opposition involution' of the corresponding diagram is trivial, that is when G is of type A_1, B_l, C_l, D_l (l even), E_7, E_8, F_4, G_2. In these cases we get $C_G(A) = A$, a maximal torus of G, and

$$M = C_K(A) = K \cap A = \{a \in A \mid \gamma a \gamma^{-1} = a^{-1} = a\}$$

consists of the 2-torsion points of A, i.e.

$$\text{(20)} \qquad M \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^l, \quad l = \operatorname{rank}(G).$$

In the other cases, A_l ($l \geqslant 2$), D_l (l odd), E_6, we have $C_G(A) = T$, $A \nsubseteq T$, where T is a maximal torus of G, and $M = C_K(A) = K \cap T$ is of the form

$$\text{(21)} \qquad M \cong (S^1)^a \times \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^b,$$

where $a + b = \dim A$ and $2a + b = \dim T$ (the values (a, b) are $(n, 0)$ for A_{2n}, $(n, 1)$ for A_{2n+1}, $(1, l-2)$ for D_l (l odd), and $(2, 2)$ for E_6).

3.3. In this section we shall exploit the existence of split involutions in certain groups. We let G, $H \subset G$, $\gamma \in G$, $\Gamma \subset G$, and $M = C_G(\Gamma)$ be as in 3.1. We further assume in this section that H is connected, i.e. that $X = G/H$ is simply connected. Since genera behave multiplicatively under coverings, results for non-simply connected spaces may be derived immediately.

LEMMA. *The group* Γ has a fixed *point on* $X = G/H$ *if and only if the centralizer* $C_G(H)$ *of* H *in* G *is conjugate to a subgroup of* M.

Proof. A point $gH \in X$ is fixed under Γ exactly when $\Gamma \subset gHg^{-1}$, which is equivalent to $M = C_G(\Gamma) \supset gC_G(H)g^{-1}$.

THEOREM. *Let G be simple. Assume that G has a split involution, i.e. that G is of type A_1, B_l, C_l, D_l (l even), E_7, E_8, F_4, G_2, and that $C_G(H)$ is not an elementary abelian 2-group. Then the signature* $\operatorname{sign}(X)$ *of the homogeneous space $X = G/H$ vanishes. If in addition $w_2(X) = 0$, then* $\Phi(X) = \operatorname{sign}(X) = 0$.

Proof. Let $\gamma \in G$ be a split involution in G. Then, in the earlier notations,

$$\operatorname{sign}(X) = \operatorname{sign}(X^\Gamma)$$

or, if $w_2(X) = 0$,

$$\Phi(X) = \Phi(X^\Gamma)$$

(by 1.4(10), resp. Theorem 1.5, (13)). By our assumption on $C_G(H)$ and the previous lemma we get $X^\Gamma = \varnothing$, thus our assertion.

We have some special cases:

COROLLARY 1. *Let G be as in the theorem and assume that $X = G/H$ admits a homogeneous complex structure. Then* $\mathrm{sign}(X) = 0$. *If in addition* $w_2(X) = 0$, *then* $\Phi(X) = 0$.

Proof. According to a theorem of H. C. Wang (cf. [5, Prop. 13.5]) H is the centralizer of a torus, now.

REMARK. If X is hermitian symmetric, then this result agrees with those in [14, p. 163]. The computations there as well as our later results show that the existence of a split involution in G is crucial.

COROLLARY 2. *Assume that G is simple, that $H \subset G$ is maximal of maximal rank, and that $X = G/H$ is not symmetric. Then* $\Phi(X) = \mathrm{sign}(X) = 0$.

Proof. According to the classification of maximal subgroups H with $\mathrm{rank}(H) = \mathrm{rank}(G)$ by Borel and de Siebenthal [8, §7], G is now of exceptional type and H is the connected centralizer of an element of order 3 or 5. Checking through the tables of *loc. cit.* and using the criteria of 2.6 one easily verifies $w_2(X) = 0$ in all cases. Moreover, the only case not covered by our theorem is that of $G = E_6$, $H = A_2 \times A_2 \times A_2$. But here $\dim G/H = 54$ which also leads to $\Phi(X) = \mathrm{sign}(X) = 0$.

REMARK. The spaces X in Corollary 2 where our methods are effectively needed, i.e. where $\dim X$ is divisible by 8, are E_8/A_8 and $E_8/A_4 \times A_4$ of dimension 168, resp. 200. The space $F_4/A_2 \times A_2$ is of dimension 36 and thus covered by our Theorem 2.3. In the remaining cases we have $\dim X \equiv 2(4)$.

3.4. Our last section dealt with cases where X^Γ is empty. Since X^Γ is always a finite union of M-orbits, it is clear that X^Γ consists of at most a finite number of points whenever M is finite, i.e. when M belongs to a split involution $\gamma \in G$. However, we shall show that this is still true for quasi-split involutions. We keep the previous notations and assumptions (in particular, H is connected and $\mathrm{rank}(H) = \mathrm{rank}(G)$).

LEMMA. *The connected centralizer $C_G(M)^0$ of $M = C_G(\Gamma)$ in G acts trivially on X^Γ.*

Proof. Let $C = C_H(H)$ denote the centre of H. Since $\mathrm{rank}(H) = \mathrm{rank}(G)$ we have $H = C_G(C)^0$ by [8, Theorème 5]. Let $gH \in X$ be fixed under Γ, thus

$$\Gamma \subset gHg^{-1}$$

or

$$M = C_G(\Gamma) \supset gC_G(H)g^{-1} \supset gCg^{-1}.$$

Centralizing once more gives

$$C_G(M) \subset gC_G(C)g^{-1}$$

hence

$$C_G(M)^0 \subset gC_G(C)^0 g^{-1} = gHg^{-1}.$$

But this means that $gH \in X$ is fixed under $C_G(M)^0$.

PROPOSITION. *Let $\gamma \in G$ be a quasi-split involution, $A \subset G$ a maximal γ-split torus and $T = C_G(A)$ its centralizer. Then the generic self-intersection $X^\Gamma = X^\gamma \circ X^\gamma$ is contained in X^T. In particular, X^Γ consists of at most finitely many points.*

Proof. By definition, T is a maximal torus of G. Since $M = K \cap T \subset T$ we have $T \subset C_G(M)^0$. By the lemma we get

$$X^\Gamma \subset X^T,$$

and by 2.5, Proposition 2, these sets are finite.

There is no loss in generality by assuming that the isotropy group H of $X = G/H$ contains T. Then

$$X^T = \{wH \mid w \in W\}$$

by 2.5, Proposition 2, and by Lemma 3.3 (proof) we have

$$X^\Gamma = \{wH \mid C_G(H) \subset w^{-1}Mw\}$$

(since T acts trivially on $M \subset T$, the notation $w^{-1}Mw$ is independent of the representative of w in $N_G(T)$).

We distinguish two cases, again:

(1) if γ is split, then M consists of all 2-torsion points of T and is stable under W. Hence we have either $X^\Gamma = \varnothing$ or $X^\Gamma = X^T$. The situation where $X^\Gamma = X^T$ is related to our approach to the signature formula in 2.5 by means of a regular S^1-action on X with fixed point set $X^{S^1} = X^T$. By choosing suitable representatives for γ and a generic conjugate $s\gamma s^{-1}$ (i.e. γ as a lift of $-1 \in W$ to $N_G(T)$ and $s \in T$ such that s^2 is regular) one can determine the canonical orientation of a point $w^{-1}H$ as component of the self-intersection $X^\Gamma = X^\gamma \circ X^\gamma$ (cf. 1.4) and compare it with the global orientation (i.e. the orientation on $T_{w^{-1}H}X$ given by the orientation of X). It turns out that both differ by the factor $(-1)^{\mu(w)}$ (or its negative, if the other global orientation is chosen). Thus one obtains a second proof (in fact, our original one) of Theorem 2.5, in this case at least. In view of the proof of formula 1.4(10) in [18], which avoids the use of the Atiyah–Bott–Singer index theorem, this derivation may be called elementary.

(2) If γ is not split (but quasi-split) the group $M = T \cap K$ is stable only under the subgroup

$$W^\gamma = \{w \in W \mid \mathrm{Ad}(\gamma)w = w\,\mathrm{Ad}(\gamma) \text{ on } T\}$$

which may be identified with the 'little' Weyl group $N_K(A)/C_K(A)$ of the symmetric space G/K (cf. [13, VII, 8.10] and use that $W(m_0)$ of *loc. cit.* is trivial since γ is quasi-split). Obviously, the condition

$$(22) \qquad C_G(H) \subset w^{-1}Mw$$

is a condition on the double cosets $W^\gamma w W(H)$ only. In general, X^Γ will have less points than X^T, however, we do not know *a priori* how many double cosets actually satisfy (22). In the following examples there will be only one, or none.

EXAMPLES. (1) Let us consider the Grassmannian $X = G_k(\mathbb{C}^n)$ of complex k-planes in $\mathbb{C}^n$ which is homogeneous under the projective unitary group $\mathrm{PU}(n)$ of type A_{n-1}. Since X is hermitian symmetric its signature can be computed by Hodge theory (see [14, p. 163]). By the same reason, our methods developed above simplify since all self intersection points in $X^\Gamma = X^\gamma \circ X^\gamma$ are positively oriented, now (X^γ is a complex submanifold of X). Hence $\mathrm{sign}(X)$ equals the cardinality of X^Γ.

For convenience, let us work with $\mathrm{U}(n)$ instead of $\mathrm{PU}(n)$ (their differences can be neglected in this context), and let $T \subset \mathrm{U}(n)$ denote the standard maximal torus consisting of all diagonal matrices

$$\delta(a_1, \ldots, a_n), \quad a_i \in S^1.$$

The element

$$\gamma = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}$$

is a quasi-split involution of $\mathrm{U}(n)$ and a maximal γ-split torus is

$$A = \{\delta(a_1, \ldots, a_m, (1), a_m^{-1}, \ldots, a_1^{-1})\}$$

(here $n = 2m$ or $2m + 1$). We have $C_G(A) = T$ and

$$M = \{t \in T \mid \gamma t = t\gamma\} = \{\delta(a_1, \ldots, a_m, (a_{m+1}), a_m, \ldots, a_1)\}.$$

Let $H = \mathrm{U}(k) \times \mathrm{U}(n-k)$ be the isotropy group of $G = \mathrm{U}(n)$ on $X = \mathrm{Gr}_k(\mathbb{C}^n)$.

Then

$$C_G(H) = \{\delta(\underbrace{a, \ldots, a}_{k}, \underbrace{b, \ldots, b}_{n-k}) \mid a, b \in S^1\},$$

and we have

$$wC_G(H)w^{-1} \subset M$$

for some $w \in W(G) = S_n$ if and only if at least one of k or $n-k$ is even. Thus $\operatorname{sign}(X) = 0$ if n is even and k is odd. Assume now, without loss of generality, k to be even, $k = 2s$, and H to be replaced by a conjugate such that

$$C_G(H) = \{\delta(\underbrace{a, \dots, a}_{s}, \underbrace{b, \dots, b}_{n-2s}, \underbrace{a, \dots, a}_{s})\}.$$

We shall identify $X^T = \{wH \mid w \in W\}$ with the quotient $W/W(H) = S_n/S_k \times S_{n-k}$ and consider the map

$$W/W(H) \xrightarrow{\quad c \quad} \{wC_G(H)w^{-1} \mid w \in W\}.$$

if $k \neq n-k$ this is a bijection, whereas for $k = n-k$ it is two-to-one. Assume $k \neq n-k$ first. Then $wC_G(H)w^{-1} \subset M$ if and only if w is of the form $w_1 w_2$ with $w_2 \in W(H)$ and $w_1 \in W^\gamma \cong S_m$ (acting by permutation on the first m coordinates and by the reflected permutation on the last m coordinates of T). Accordingly, there are $\binom{m}{s}$ different W-conjugates of $C_G(H)$ in M. If $k = n-k$ we only get half the number of conjugates but this is made up for by c being $2:1$, so that

$$\text{(23)} \qquad \operatorname{sign}(\mathrm{Gr}_k(\mathbb{C}^n)) = \begin{cases} 0 & n \text{ even, } k \text{ odd} \\ \dbinom{m}{s} & \begin{array}{l} n = 2m \text{ or } 2m+1 \\ 2s = k \text{ or } n-k \end{array} \end{cases}$$

which agrees with [14, p. 163].

(2) In a similar way we can deal with the hermitian symmetric spaces for G of type D_{2m+1} and E_6, which are

$$\mathbf{Q}_{4m} = \mathrm{SO}(4m+2)/\mathrm{SO}(4m) \times \mathrm{SO}(2)$$
$$\text{(complex quadric, } \dim_{\mathbb{C}} = 4m)$$

$$\mathbf{F}_{2m+1} = \mathrm{SO}(4m+2)/\mathrm{U}(2m+1)$$
$$(\dim_{\mathbb{C}} = m(2m+1))$$

$$\mathbf{W}_{\mathbb{C}} = E_6/\mathrm{Spin}(10) \cdot S^1$$
$$\text{(complexified Cayley plane, } \dim_{\mathbb{C}} = 16).$$

If we realize these spaces in the form G/H, G adjoint, H connected, then $C_G(H) \cong S^1$ and $H = C_G(C_G(H))$. We also have $N_G(H) = N_G(C_G(H)) = H$ except for $\mathbf{Q}_{4m}$ where H is of index 2 in $N_G(H)$.

Fix a maximal torus T of G. Then a quasi-split involution $\gamma \in N_G(T)$ can be obtained as a lift of $w_0 \in N_G(T)/T$ (w_0 is the longest element in W with respect to some basis of simple roots; we have $w_0 = -\iota = c^{h/2}$, where ι is the opposition involution, c a Coxeter element, h the Coxeter number). With this choice T contains a maximal γ-split torus A (of rank $2m$ resp. 4 for type D_{2m+1} resp. E_6) such that $C_G(A) = T$ and

$$M = T^\gamma \cong \begin{cases} \left(\dfrac{\mathbb{Z}}{2\mathbb{Z}}\right)^{2m-1} \times S^1 & (D_{2m+1}) \\ \left(\dfrac{\mathbb{Z}}{2\mathbb{Z}}\right)^{2} \times S^1 \times S^1 & (E_6) \end{cases}$$

With the help of the affine coordinate diagram (attached to G/H, now) it is seen that M contains one (resp. no, resp. 3) conjugate(s) of $C_G(M) \cong S^1$ in the case of $\mathbf{Q}_{4m}$ (resp. $\mathbf{F}_{2m+1}$, resp. $\mathbf{W}_{\mathbb{C}}$). Due to $|N_G(H)/H| = 2$ in the first case, we get

$$\operatorname{sign}(\mathbf{Q}_{4m}) = \Phi(\mathbf{Q}_{4m}) = 2,$$

$$\operatorname{sign}(\mathbf{F}_{2m+1}) = \Phi(\mathbf{F}_{2m+1}) = 0,$$

$$\operatorname{sign}(\mathbf{W}_{\mathbb{C}}) = \Phi(\mathbf{W}_{\mathbb{C}}) = 3,$$

which, again, may be found in [14, p. 163] (for sign).

REMARKS. (1) Another way to deduce the signature of $X = \mathrm{Gr}_k(\mathbb{C}^n)$ from formula (10) in 1.4 is the following. Let σ denote the usual complex conjugation on X (which is not homotopic to the identity). Then X^σ is the real Grassmannian $\mathrm{Gr}_k(\mathbb{R}^n)$ of which X is the complexification. Accordingly, the normal bundle of X^σ in X is isomorphic to the (real) tangent bundle of X^σ. This gives

$$\operatorname{sign}(X^\sigma \circ X^\sigma) = (-1)^{d/2} \cdot e(X^\sigma),$$

where $d = \dim_{\mathbb{C}} X$ (note that we have kept the usual complex orientation on X, compare [18, §3]). Since all homology classes of X can be represented by algebraic cycles defined over $\mathbb{R}$, the involution σ acts on $H_d(X, \mathbb{R})$ by multiplication with $(-1)^{d/2}$. Thus

$$\operatorname{sign}(X) = (-1)^{d/2} \operatorname{sign}(X)_\sigma,$$

hence by (10)

$$\operatorname{sign}(X) = e(X^\sigma).$$

The same reasoning works in the case of $\mathbf{Q}_{4m}$ and $\mathbf{W}_{\mathbb{C}}$. In the first case one has to choose a complex conjugation associated to a split real form. In the second case the real form is the Cayley plane $\mathbf{W} = F_4/\mathrm{Spin}(9)$ with $e(\mathbf{W}) = 3$.

(2) Along the lines of the examples above one may also attack the non-

TABLE I

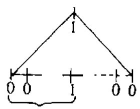
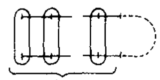
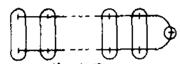
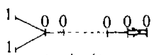
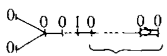
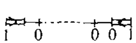
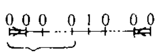
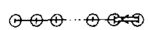
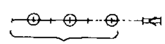
$\mathfrak{g}$	Affine coordinate diagram	$\mathfrak{k}$	$\dim \tilde{S}$	Index diagram	$\mathfrak{c}$	$\mathfrak{m}$	Spin	$e(\tilde{S})$	$\operatorname{sign}(\tilde{S})$
A_1		$\mathbb{R}$	2	$\oplus$	$\mathbb{R}$	0	$+$	2	0
$A_l\ (l>1)$ $(\mathrm{su}(l+1))$	 $(1 \leq p \leq (l+1)/2)$	$A_{p-1} \oplus A_{l-p} \oplus \mathbb{R}$ $\begin{pmatrix} \mathrm{su}(p) \oplus \mathrm{su}(q) \oplus \mathbb{R} \\ p+q=l+1 \end{pmatrix}$	$2pq$	 $p<(l+1)/2$  $p=(l+1)/2$	$A_{l-2p} \oplus \mathbb{R}^{2P}$ $\mathbb{R}^l$	$A_{2l-2p} \oplus \mathbb{R}^P$ $\mathbb{R}^P$	$+(l \text{ odd})$ $-(l \text{ even})$	$\binom{l+1}{P}$	see 3.4
$B_l\ (l>1)$ $(\mathrm{so}(2l+1))$	 $(k=l-1)$  $(0 \leq k < l-1)$	$B_k \oplus D_{l-k}$ $\begin{pmatrix} \mathrm{so}(p) \oplus \mathrm{so}(q) \\ p=2k+1 \\ q=2(l-k) \end{pmatrix}$	pq	 $r=\min(p,q)$	$B_{l-r} \oplus \mathbb{R}^r$	B_{l-r} $(B_0=0)$	$+\ (k=0)$ $-(k>0)$	$2\binom{l}{k}$	0 ?
$C_l\ (l>2)$ $(\mathrm{sp}(l))$	  $(1 \leq p \leq l/2)$	$A_{l-1} \oplus \mathbb{R}$ $(\mathrm{u}(l))$ $C_p \oplus C_{l-p}$ $\begin{pmatrix} \mathrm{sp}(p) \oplus \mathrm{sp}(q) \\ p+q=l \end{pmatrix}$	$l(l+1)$ $4pq$	  $2p$	$\mathbb{R}^l$ $C_{l-2p} \oplus A_1^p \oplus \mathbb{R}^p$	0 $C_{l-2p} \oplus A_1^p$	$+\ (l \text{ odd})$ $-\ (l \text{ even})$ $+$	2^l $\binom{l}{p}$	0 ?

TABLE I (*continued*)

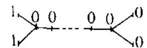
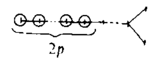
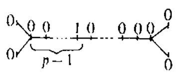
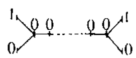
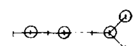
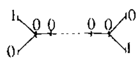
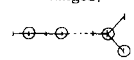
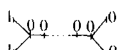
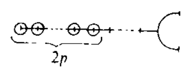
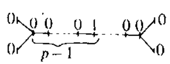
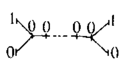
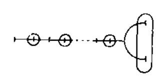
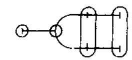
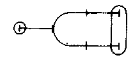
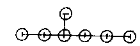
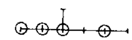
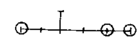
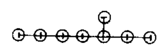
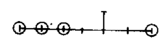
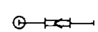
$\mathfrak{g}$	Affine coordinate diagram	$\mathfrak{k}$	$\dim \tilde{S}$	Index diagram	$\mathfrak{c}$	$\mathfrak{m}$	Spin	$e(\tilde{S})$	$\operatorname{sign}(\tilde{S})$
D_{2m}	 $(p=1)$	$D_p \oplus D_{2m-p}$	$4pq$	 $(1 \leqq p \leqq m)$	$D_{2m-2p} \oplus \mathbb{R}^{2p}$	D_{2m-2p}	$+$	$2\binom{2m}{p}$	0
$(m \geqq 2)$ $(\mathrm{so}(4m))$	 $(2 \leqq p \leqq m)$	$\begin{pmatrix} \mathrm{so}(2p) \oplus \mathrm{so}(2q) \\ p+q=2m \end{pmatrix}$				$(D_0 = 0)$			?
		$A_{2m-1} \oplus \mathbb{R}$ $(\mathrm{u}(2m))$	$2m(2m-1)$	 (m even, for m odd both diagrams are interchanged)	$A_1^m \oplus \mathbb{R}^m$	A_1^m	$+$	2^{2m-1}	0
		$A_{2m-1} \oplus \mathbb{R}$	$2m(2m-1)$						
D_{2m+1} $(m \geqq 2)$ $(\mathrm{so}(4m+2))$	 $p=1$	$D_p \oplus D_{2m+1-p}$ $\begin{pmatrix} \mathrm{so}(2p) \oplus \mathrm{so}(2q) \\ p+q=2m+1 \end{pmatrix}$	$4pq$	 $(1 \leqq p \leqq m)$	$D_{2m-2p+1} \oplus \mathbb{R}^{2p}$	$D_{2m-2p+1}$ $(D_1 \cong \mathbb{R}!)$	$+$	$2\binom{2m+1}{p}$	2
	 $(2 \leqq p \leqq m)$								?
		$A_{2m} \oplus \mathbb{R}$ $(\mathrm{u}(2m+1))$	$2m(2m+1)$		$A_1^m \oplus \mathbb{R}^{m+1}$	$A_1^m \oplus \mathbb{R}$	$+$	2^{2m}	0

TABLE I (*continued*)

$\mathfrak{g}$	Affine coordinate diagram	$\mathfrak{k}$	$\dim \tilde{S}$	index diagram	$\mathfrak{c}$	$\mathfrak{m}$	Spin	$e(\tilde{S})$	$\mathrm{sign}(\tilde{S})$	b_m
E_6	0 0 0 0 0, 1, 0	$A_5 \oplus A_1$	40		$\mathbb{R}^6$	$\mathbb{R}^2$	+	36	4	4
	1 0 0 0 1, 0, 0	$D_5 \oplus \mathbb{R}$	32		$A_3 \oplus \mathbb{R}^2$	A_3	+	27	3	3
E_7	0 0 0 0 0 0 0, 1	A_7	70		$\mathbb{R}^7$	0	+	72	0	0
	0 1 0 0 0 0 0, 0	$D_6 \oplus A_1$	64		$A_1^3 \oplus \mathbb{R}^4$	A_1^3	+	63	7	7
	1 0 0 0 0 0 0, 0	$E_6 \oplus \mathbb{R}$	54		$D_4 \oplus \mathbb{R}^3$	D_4	+	56	0	0
E_8	0 0 0 0 0 0 0 1, 0	D_8	128		$\mathbb{R}^8$	0	+	135	7	9
	0 1 0 0 0 0 0 0	$E_7 \oplus A_1$	112		$D_4 \oplus \mathbb{R}^4$	D_4	+	120	8	8
F_4	0 0 0 1 0	$C_3 \oplus A_1$	28		$\mathbb{R}^4$	0	−	12	0	0
	1 0 0 0 0	B_4	16		$B_3 \oplus \mathbb{R}$	B_3	+	3	1	1
G_2	0 1 0	$A_1 \oplus A_1$	8		$\mathbb{R}^2$	0	+	3	1	1

hermitian symmetric spaces of type D_{2m+1} and E_6. In these cases X^Γ is always reduced to an orbit under the 'small' Weyl group W^γ ($\cong W(B_{2m}), W(F_4)$) and the signature can be expressed as a sum of terms $(-1)^{\mu(w)}$, w ranging over representatives of this orbit (e.g. for $X = E_6/A_5 \times A_1$ there are 12 terms in contrast to the formula of Theorem 2.5 which yields 36).

(3) In the above developments (from 3.2 on) we have concentrated on quasi-split involutions γ only, the advantage being the discreteness of X^Γ. On the other hand, the associated orientation sums are hard to compute (in case X is not complex) and one would like to cluster together groups of points of X^Γ to sum in two steps. In a sense this is achieved by considering the self-intersection sets $X^\gamma \circ X^\gamma$ for involutions γ with 'big' group M. The result of the first summing process is then given by the signature of the components of $X^\gamma \circ X^\gamma$ (which are homogeneous under M). We have studied a number of such involutions, in particular the natural symmetries on symmetric spaces (where X^γ is an antipodal set). Here, the determination of the components of X^γ and $X^\gamma \circ X^\gamma$ requires quite detailed and specific information on conjugacy classes and centralizers in G. The problem of computing orientation numbers remains, and, at this moment, we have not got beyond the known cases. However, we hope to complete these investigations (with which we originally started) and to report about them at some other occasion.

4. Tables

In Table I we have listed the conjugacy classes of involutions in simple (adjoint) groups. The first six columns are explained in 3.1. The + (resp. −) in the 'spin' column indicates the existence (resp. non-existence) of a spin structure on the simply connected symmetric space $\tilde{S} = G/K^0$. Column 8 gives the Euler number $e(\tilde{S}) = |W(G)/W(K^0)|$ of $\tilde{S}$, column 9 the signature as far as we know it (in the case of the real and quaternionic Grassmannians we have refrained from inserting the value for obvious cases, like spheres, projective spaces, spin manifolds of dimension $\not\equiv 0$ (8), etc.).

The signatures of the spaces $E_6/A_5 \times A_1$, $E_7/D_6 \times A_1$, $E_8/E_7 \times A_1$ and E_8/D_8 have been computed by Bliss, Moody and Pianzola, cf. their appendix to this paper [7].

For the exceptional symmetric spaces we have also listed the middle Betti number b_m, $m = \frac{1}{2}\dim \tilde{S}$. One notes that the intersection form on $H_m(\tilde{S}, \mathbb{Q})$ is indefinite only if $\tilde{S} = E_8/D_8$. In the definite (exceptional) cases the intersection form is the standard diagonal form except for $E_8/E_7 \times A_1$ where it might also be the form of type E_8. We leave it as a problem to settle this question.

REFERENCES

1. *Proc. Conf. on Elliptic Curves and Modular Forms in Algebraic Topology, Princeton, September 1986*; *Lecture Notes in Maths* **1326**, Springer-Verlag, Berlin, Heidelberg, New York, 1988.
2. Adams, J. F., *Lectures on Lie Groups*, Benjamin, New York, 1969.
3. Atiyah, M. F., and Hirzebruch, F., 'Spin-manifolds and group actions', in *Memoires dédiés à Georges de Rham*, Springer-Verlag, Berlin, Heidelberg, New York, 1970, pp. 18–28.
4. Atiyah, M. F., and Singer, I. M., 'The index of elliptic operators III', *Ann. Math.* **87** (1966), 546–604.
5. Borel, A. and Hirzebruch, F., 'Characteristic classes and homogeneous spaces I', *Amer. J. Math.* **80** (1958), 458–538.
6. Borel, A. and Hirzebruch, F., 'Characteristic classes and homogeneous spaces II', *Amer. J. Math.* **81** (1959), 315–382.
7. Bliss, J. G., Moody, R. V., and Pianzola, A., Appendix to this article, *Geom. Dedicata* **35** (1990), 345–351 (this issue).
8. Borel, A. and de Siebenthal, J., 'Les sous-groupes fermés de rang maximum des groupes de Lie clos', *Comment Math. Helv.* **23** (1949), 200–221.
9. Bott, R., 'Vector fields and characteristic numbers', *Michigan Math. J.* **14** (1967), 231–244.
10. Bott, R. and Taubes, C., 'On the rigidity theorem of Witten', *J. Amer. Math. Soc.* **2** (1989), 137–186.
11. Bourbaki, N., *Groupes et algèbres de Lie*, Chap. IV, V, VI. Hermann, Paris, 1968.
12. Golubitsky, M. and Guillemin, V., *Stable Mappings and Their Singularities*, Springer-Verlag, Berlin, Heidelberg, New York, 1973.
13. Helgason, S., *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.
14. Hirzebruch, F., *Topological Methods in Algebraic Geometry* (3rd edn, with appendices by R. L. E. Schwarzenberger and A. Borel), Springer-Verlag, Berlin, Heidelberg, New York, 1966.
15. Hirzebruch, F., 'Involutionen auf Mannigfaltigkeiten' in *Proc. Conf. on Transformation Groups, New Orleans 1967*, Springer-Verlag, Berlin, Heidelberg, New York, 1968, pp. 148–166.
16. Hirzebruch, F., 'Elliptic genera of level *N* for complex manifolds', in *Differential Geometrical Methods in Theoretical Physics*, Kluwer, Dordrecht, 1988, pp. 37–63.
17. Hopf, H. and Samelson, H., 'Ein Satz über die Wirkungsräume geschlossener Liescher Gruppen', *Comment. Math. Helv.* **13** (1940/41), 240–251.
18. Jänich, K. and Ossa, E., 'On the signature of an involution', *Topology* **8** (1969), 27–30.
19. Kleiman, S. L., 'The transversality of a general translate', *Comp. Math.* **38** (1974), 287–297.
20. Landweber, P. S., 'Elliptic genera: An introductory overview', in reference [1], pp. 1–10.
21. Milnor, J., 'Spin structures on manifolds', *L'Enseignement Math.* **9** (1963), 198–203.
22. Ochanine, S., 'Sur les genres multiplicatifs définis par des intégrales elliptiques', *Topology* **26** (1987), 143–151.
23. Ochanine, S., 'Genres elliptiques equivariants', in [1], p. 107–122.
24. Selbach, M., *Klassifikationstheorie halbeinfacher algebraischer Gruppen*, Bonner Mathematische Schriften **83**, Bonn, 1976.
25. Springer, T. A., 'The classification of involutions of simple algebraic groups', *Journal Fac. Sci. University of Tokyo, Sec. IA*, **34** (1987), 655–670.
26. Tits, J., 'Sur la classification des groupes algébriques simples', *C.R.A.S. Paris* **249** (1959), 1438–1440.
27. Tits, J., 'Classification of algebraic semisimple groups', *Proc. Symp. Pure Math.* **9** (1966), 33–62.
28. Witten, E., 'The index of the Dirac operator in loop space', in [1], p. 161–181.
29. Zagier, D., 'Note on the Landweber-Stong Elliptic genus', in [1], p. 216–224.

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