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THE TOPOLOGY OF NORMAL SINGULARITIES OF AN ALGEBRAIC SURFACE

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(d'après un article de D. MUMFORD [4])

We shall study MUMFORD's results in the complex-analytic case.

1. Regular graphs of curves.

Let X be a complex manifold of complex dimension 2. A regular graph Γ of curves on X is defined as follows.

i. $\Gamma = \{E_1, E_2, \dots, E_n\}$.

ii. Each E_i is a compact connected complex submanifold of X of complex dimension 1.

iii. Each point of X lies on at most two of the E_i.

iv. If $x \in E_i \cap E_j$ and $i \neq j$, then E_i, E_j intersect regularly in x and $E_i \cap E_j = \{x\}$.

Γ defines a graph Γ' in the usual sense (i. e. a one-dimensional finite simplicial complex) by associating to each E_i a vertex e_i and by joining e_i and e_j by an edge if and only if $E_i \cap E_j$ intersect. Γ' becomes a "weighted graph" by attaching to each e_i the self-intersection number $E_i . E_i$, i. e. the Euler number of the normal bundle of E_i in X. We have the symmetric matrix

$$S(\Gamma) = ((E_i . E_j))$$

where $E_i . E_j$ $(i \neq j)$ equals 1 if $E_i \cap E_j \neq \emptyset$ and equals 0 if $E_i \cap E_j = \emptyset$. This matrix is called the intersection matrix of Γ and defines a bilinear symmetric form S over the $\mathbb{Z}$-module $V = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \dots + \mathbb{Z}e_n$. The matrix $S(\Gamma)$ depends (up to the ordering of the e_i) only on the weighted tree and may be denoted by $S(\Gamma')$. The subset A of X is called a tubular neighbourhood of Γ if

i. $A = \bigcup_{i=1}^{n} A_i$,

where A_i is a (compact) tubular neighbourhood of E_i,

ii. $E_i \cap E_j = \emptyset$ implies $A_i \cap A_j = \emptyset$

iii. $E_i \cap E_j = \{x\}$ implies the existence of a local coordinate system (z_1 , z_2) with center x and a positive number ε such that the open neighbourhood

$$U = \{p| \quad p \in X \wedge |z_1(p)| < 2\varepsilon \wedge |z_2(p)| < 2\varepsilon\}$$

is defined in this coordinate system and

$$A_i \cap U = \{p| \quad p \in U \cap |z_2(p)| \leq \varepsilon\} \quad ,$$

$$A_j \cap U = \{p| \quad p \in U \cap |z_1(p)| \leq \varepsilon\} \quad ,$$

$$A_i \cap A_j \subset U \quad .$$

Such tubular neighbourhoods always exist.

A is a compact 4-dimensional manifold (differentiable except "corners") whose boundary M is a 3-dimensional manifold (without boundary). It is easy to see that A has $E = \bigcup_{i=1}^{n} E_i$ as deformation retract. Thus

(1) $$H_i(A) \sim H_i(E) \quad .$$

Suppose that the graph Γ' is connected. This is the case if M is connected. If, moreover, Γ' has no cycles, then E is homotopically equivalent to a wedge of n compact oriented topological surfaces with the genera $g_i = \text{genus}(E_i)$. If Γ' has p linearly independent cycles, then the homotopy type of E is the wedge of n surfaces as above and p one-dimensional spheres. The first Betti number of E is given by the formula

(2) $$b_1(E) = 2 \sum_{i=1}^{n} g_i + p \quad .$$

We have the exact sequence (rational cohomology)

(3) $$H^1(A , M) \to H^1(A) \to H^1(M) \quad .$$

By Poincaré duality $H^1(A , M) \cong H_3(A)$ which vanishes by (1).

Therefore $H^1(A)$ maps injectively into $H^1(M)$ which proves in virtue of (1) and (2) :

LEMMA. - <u>If the regular graph of curves</u> $\Gamma = \{E_1 , \cdots , E_n\}$ <u>has a tubular neighbourhood</u> A <u>whose boundary</u> M <u>is a rational homology sphere, then the graph</u> Γ' <u>is a tree (i. e.</u> Γ' <u>is connected and has no cycles). Furthermore, the genera of the curves are all</u> 0 , <u>thus all the</u> E_i <u>are 2-spheres.</u>

2. The fundamental group of the "tree manifold" M.

Suppose M is obtained as in Section 1, assume that Γ' is a tree and all the E_i are 2-spheres. By the lemma of Section 1 this is true if M is a rational homology sphere. The fundamental group $\pi_1(M)$ is presented by the following theorem.

THEOREM. – Put $S(\Gamma) = ((E_i.E_j)) = s_{ij}$. Then, with the above assumptions, $\pi_1(M)$ is isomorphic with the free group generated by the vertices $e_1, \dots, e_n$ of Γ' modulo the relations

(a) $$e_i\, e_j^{s_{ij}} = e_j^{s_{ij}}\, e_i$$

(b) $$1 = \prod_{1 \leqslant j \leqslant n} e_j^{s_{ij}} ,$$

the product in (b) being ordered from left to right by increasing j. Recall that the exponents s_{ij} are all 1 or 0 (for $i \neq j$).

Remark. – Each weighted tree with a numbering of its vertices defines by this recipe a group. A change of the numbering gives an isomorphic group. This is not difficult to prove. Thus it makes sense to speak (up to an isomorphism) of $\pi_1(\Gamma')$ where Γ' is any weighted tree.

We sketch a proof of the theorem. The boundary of A_i, denoted by ∂A_i, is a circle bundle over S^2 with Euler number s_{ii}. A generator e_i of $\pi_1(\partial A_i)$ is represented by a fibre. The only relation is

$$e_i^{s_{ii}} = 1 .$$

Recall $M = \partial A$ and put $B_i = \partial A \cap A_i$ which is a 3-dimensional manifold obtained from ∂A_i by removing for each j with $j \neq i$ and $s_{ij} \neq 0$ a fibre preserving neighbourhood of some fibre This neighbourhood to be removed has in local coordinates (Section 1, (iii)) the description $(|z_1| < \varepsilon ,\ |z_2| = \varepsilon)$ and thus is of the type $D^2 \times S^1$. The boundary of B_i consists of a certain number of 2-dimensional tori (one for each j with $j \neq i$ and $s_{ij} \neq 0$). The fundamental group $\pi_1(B_i)$ has generators e_j $(j = i$ or $s_{ij} \neq 0)$ with the only relations

(a) $$e_i\, e_j = e_j\, e_i$$

(b) $$e_i^{-s_{ii}} = \prod e_j ,$$

the product is in increasing order of j (over those e_j with $j \neq i$ and $s_{ij} \neq 0$). Here e_i is representable by any fibre, thus also by a fibre on the

j^{th} torus. e_j is represented on the j^{th} torus by ($z_1 = \varepsilon^{2\pi it}$, $z_2 =$ constant of absolute value 1). It becomes a fibre in B_j . Since $M = \bigcup B_i$, we can use van Kampen's theorem to present $\pi_1(M)$ as the free product of the $\pi_1(B_i)$ modulo amalgamation of certain subgroups $\pi_1(S^1 \times S^1)$. This gives the theorem. Our notation takes automatically care of the amalgamation because for $s_{ij} \neq 0$ and $i \neq j$ the symbols e_i , e_j denote elements of $\pi_1(B_i)$ and of $\pi_1(B_j)$. Of course, there is all the trouble with the base point which we have neglected in this sketch. The trouble is not serious, mainly because Γ' is a tree. A further remark to visualize the relations : B_i , as a circle bundle over S^2 - (disjoint union of small disks), is trivial. Thus e_i lies in the center of $\pi_1(B_i)$. There is a section of ∂A_i over the oriented S^2 with one singular point. This gives an "oriented disk-like 2-chain" in ∂A_i with $e_i^{-s_{ii}}$ as boundary (characteristic class = negative transgression!). The small disks lift to disks in that 2-chain. They have to be removed and have the e_j $(j \neq i\ ,\ s_{ij} \neq 0)$ as boundary. Knowledge of the fundamental group of a disk with small disks removed gives (b).

COROLLARY. - The determinant of the matrix (s_{ij}) is different from 0 if and only if $H_1(M\ ;\ \mathbb{Z})$ is finite. If this is so, then $|\det(s_{ij})|$ equals the order of $H_1(M\ ;\ \mathbb{Z})$.

Proof. - Recall that $H_1(M\ ;\ \mathbb{Z})$ is the abelianized $\pi_1(M)$. The corollary follows from relation (b) of the theorem. The result can also be obtained directly from the exact homology sequence of the pair $(A\ ,\ M)$ which identifies $H_1(M\ ;\ \mathbb{Z})$ with the cokernel of the homomorphism $V \to V^*$ defined by the quadratic form S (for the notation see Section 1). $H_2(A\ ;\ \mathbb{Z})$ may be identified with V and $H_2(A\ ,\ M\ ;\ \mathbb{Z})$ by Poincaré duality with $V^* = \mathrm{Hom}(V\ ,\ \mathbb{Z})$.

3. Elementary trees.

In this section we shall prove a purely algebraic result.

A weighted tree is a finite tree with an integer associated to each vertex.

An elementary transformation (of the first kind) of a weighted tree adds a new vertex x , joins it to an old vertex y by a new edge, gives x the weight -1 and y the old weight diminished by 1 . Everything else remains unchanged.

An elementary transformation (of the second kind) adds a new vertex x , joins it to the two vertices y_1 , y_2 of an edge k by edges k_1 , k_2 , removes k , gives x the weight -1 and y_i $(i = 1, 2)$ the old weight of y_i diminished by 1. The following proposition is easy to prove.

PROPOSITION. – *If* Γ' *is a weighted tree and* Γ'' *obtainable from* Γ *by an elementary transformation, then* $S(\Gamma'')$ *is negative definite if and only if* $S(\Gamma')$ *is. Furthermore* $\pi_1(\Gamma') \sim \pi_1(\Gamma'')$ (for the notation see Section 1 and the Remark in Section 2).

An elementary tree is a weighted tree obtainable from the one-vertex-tree with weight -1 by a finite number of elementary transformations.

THEOREM. – *Let* Γ' *be a weighted tree. Suppose that* $\pi_1(\Gamma')$ *is trivial and that the matrix (integral quadratic form)* $S(\Gamma')$ *is negativ definite. Then* Γ' *is an elementary tree.*

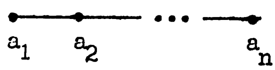
For the proof a group theoretical lemma is essential whose proof we omit.

LEMMA. – *Let* G_1, G_2, G_3 *be non-trivial groups, and* $a_i \in G_i$. *Then the free product* $G_1 * G_2 * G_3$ *modulo the relation* $a_1 a_2 a_3 = 1$ *is a non-trivial group.*

Inductive proof of the theorem. Suppose it is proved if the number of vertices in the weighted tree is less than n. Let Γ' have n vertices $e_1, \dots, e_n$.

First case. – *There is no vertex in* Γ' *which is joined by edges with at least three vertices.*

Then Γ' is linear



where a_i is the associated weight. It follows that one of the a_i must be -1, if not $\det S(\Gamma')$ would be up to sign the numerator of the continued fraction

$$|a_1| - \cfrac{1}{|a_2| - \ddots - \cfrac{1}{|a_n|}} \qquad (a_i \leq -2)$$

which is not 1. This contradicts the corollary in Section 2. Thus Γ' is an elementary transform of a tree Γ'' with $n-1$ vertices. By the proposition and the induction assumption Γ' is elementary.

Second case. – *There is a vertex* e_1, *say, joined with* $e_2, \dots, e_m$ $(m \geq 4)$.

We may choose this notation since the numbering plays no rôle for the fundamental group (see the Remark in Section 2).

Take Γ' remove e_1 and the edges joining it to $e_2, \dots, e_m$. The remaining one-dimensional complex is a union of $m-1$ trees $T_2, \dots, T_m$ where T_i has e_i as edge. The free product of the $\pi_1(T_i)$, $i = 2, \dots, m$, modulo the relation $e_2 e_3 \dots e_m = 1$ gives obviously (see Section 2) the group $\pi_1(\Gamma')$ modulo $e_1 = 1$. By assumption $\pi_1(\Gamma')$ is trivial. By the lemma at least one of the groups $\pi_1(T_i)$, say $\pi_1(T_2)$, is trivial. By induction assumption T_2 is elementary and thus can be reduced by removing a vertex x with weight -1 to give a weighted tree T_2' of which T_2 is an elementary transform of first or second kind. If $x \neq e_2$ or if $x = e_2$ and joined only with one vertex in T_2, then Γ' is elementary transform of the tree consisting of the T_i, $(i = 3, \dots, m)$, T_2', and e_1 (with the weight unchanged or increased by 1 respectively). By induction and the proposition, Γ' would be elementary. In the remaining case $x = e_2$ and e_2 is joined with exactly three vertices in Γ', namely e_1 and, say, e_{m+1}, e_{m+2} of T_2. Again, either Γ' would be elementary transform of a smaller tree, or the weight of e_1 or e_{m+1} or e_{m+2} would be -1. But the latter case cannot occur, since the quadratic form takes on $e_r + e_s \in V$ (see Section 1) the value 0, if e_r, e_s have weight -1 and are joined by an edge, and this would be true for $r = 2$ and $s = 1$, $m+1$ or $m+2$ and **contradict the negative definiteness** of $S(\Gamma')$.

4. A blowing-down theorem.

THEOREM. – Let X be a complex manifold of complex dimension 2 and $\Gamma = \{E_1, E_2, \dots, E_n\}$ a regular graph of curves on X. Suppose the boundary of some tubular neighbourhood of Γ be simply-connected and the matrix $S(\Gamma')$ negative-definite. Then the topological space X/E (i. e. X with $E = \bigcup_{i=1}^{n} E_i$ collapsed to a point) is a complex manifold in a natural way: The projection $X \to X/E$ is holomorphic and the bijection $X - E \to X/E - E/E$ is biholomorphic.

Proof. – By the lemma in Section 1 and the theorem in Section 3 all curves E_i are 2-spheres and Γ' is an elementary tree. If Γ' has only one vertex, then the above theorem is due to GRAUERT or, in the classical algebraic geometric case, to CASTELNUOVO-ENRIQUES. By the very definition of an elementary tree and easy properties of "quadratic transformations" the result follows.

5. Resolution of singularities.

Let Y be a complex space of complex dimension 2 in which all points are non-singular except possibly the point y_0 which is supposed to be normal. The theorem on desingularization states that there exist a complex manifold X, a regular (see Section 1) graph Γ of curves $E_1, \dots, E_n$ on X, a holomorphic map $\pi : X \to Y$ with

$$\pi(E) = \{y_o\}, \text{ where } E = \bigcup_{i=1}^{n} E_i ,$$

$$\pi | X - E : X - E \to Y - \{y_o\} \text{ biholomorphic} .$$

Thus the topological investigation of A and M (Section 1) which we have carried through so far contains as special case the investigation of singularities. A theorem, which we do not prove here, states that $S(\Gamma)$ is negative-definite if Γ comes from desingularizing a singularity.

6. The Main theorem of Mumford.

THEOREM. – Let Y, y_o be as in Section 5. Suppose that y_o has in Y a neighbourhood U homeomorphic to R^4 by local coordinates $t_1, \dots, t_4$. Then y_o is non-singular.

"Desingularize" y_o as in Section 5. Take a tubular neighbourhood A of Γ. We can find a positive number δ such that $K = \pi^{-1}\{p \mid p \in U \wedge \sum t_i^2(p) < \delta\} \subset A$. There exists a tubular A' with

$$A' \subset K \subset A$$

and such that A' is obtained from A just by multiplying the "normal distances" by a fixed positive number $r < 1$. Any path in $A - E$ is homotopic to a path in $A' - E$ which is nullhomotopic in $A - E$ because $\pi_1(K - E) = \pi_1(R^4 - \{0\})$ is trivial. The theorem in Section 4 together with the theorem mentioned at the end of Section 5 completes the proof.

7. Further remarks.

For any weighted tree Γ' the construction in Section 1 can be topologized (assume genus $(E_i) = 0$). In this way we may attach to each weighted tree Γ' a 3-dimensional manifold $M(\Gamma')$ (see von RANDOW [5]) which, as can be shown, depends only on Γ' (up to a homeomorphism).

We have $\pi_1(M(\Gamma')) = \pi_1(\Gamma')$ (See Section 2). Von RANDOW [5] has investigated the tree manifold $M(\Gamma')$ and shown in analogy to Mumford's theorem (Section 6) that $M(\Gamma')$ is homeomorphic S^3 if $\pi_1(\Gamma')$ is trivial. Thus there is no counterexample to Poincaré's conjecture in the class of tree manifolds $M(\Gamma')$. Von Randow's investigations and also the topological part of Mumford's paper are in close connection to the classical paper of SEIFERT [6]. The oriented Seifert manifolds (fibred in circles over S^2 with a finite number of exceptional fibres) are special tree manifolds [5].

Interesting trees (always with genus $(E_i) = 0$) occur when desingularizing the singularities

$$(z_1^2 + z_2^n)^{1/2}, \quad (n \geq 2), \qquad (z_1(z_2^2 + z_1^n))^{1/2}, \quad (n \geq 2),$$
$$(z_1^3 + z_2^4)^{1/2}, \qquad (z_1(z_1^2 + z_2^3))^{1/2}, \qquad (z_1^3 + z_2^5)^{1/2}.$$

Each of these algebroid function elements generates a complex space with a singular point at the origin.

These singularities give rise to the well known trees A_{n-1}, D_{n+2}, E_6, E_7, E_8 of Lie group theory (all vertices weighted by -2). The corresponding manifolds M are homeomorphic to S^3/G where G is a finite subgroup of S^3 (cyclic, binary dihedral, binary tetrahedral, binary octahedral, binary pentagondodecahedral). Up to inner automorphisms these are the only finite subgroups of S^3. The manifold $M(E_8)$ is specially interesting. Since $\det S(E_8) = 1$, it is by the corollary in Section 2 a Poincaré manifold, i. e. a 3-dimensional manifold with non-trivial fundamental group and trivial abelianized fundamental group. $M(E_8)$ was constructed by "plumbing" 8-copies of the circle bundle over S^2 with Euler number -2. By replacing this basic constituent by the tangent bundle of S^{2k} one obtains a manifold $M^{4k-1}(E_8)$ of dimension $4k-1$. This carries a natural differentiable structure. For $k \geq 2$ it is homeomorphic to S^{4k-1}, but not diffeomorphic (Milnor sphere).

The above mentioned singularities are classical (e. g. DU VAL [1]). For the preceding remarks see also [3].

For quadratic transformations, desingularization, etc. see the papers of ZARISKI and also [2]. We have only been able to sketch some aspects of Mumford's paper, leaving others aside, e. g. the local Picard variety, etc.

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